Theoretical Optics

Exercise 1

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___/24 points
Drop point: Your tutorial group in ILIAS
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Problem 1. (10 points) Maxwell's equations in homogeneous and inhomogeneous media

Consider Maxwell's equations in a homogeneous, isotropic, dispersive, source-free, dielectric and magnetic medium characterized by a relative permittivity $\varepsilon_\mathrm{r}(\omega)$ and relative permeability $\mu_\mathrm{r}(\omega)$:

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \qquad \nabla \cdot \mathbf{B}(\mathbf{r},t) = 0,$$
$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}, \qquad \nabla \cdot \mathbf{D}(\mathbf{r},t) = 0.$$

(a) Show that if the electric field is a time harmonic plane wave characterized by a wave vector $\mathbf{k} = k\mathbf{u}$ and frequency ω, i.e. $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{-i(\omega_0 t - \mathbf{k}\cdot\mathbf{r})}$, then $\mathbf{k}$ satisfies the relations written as

$$\mathbf{k}\cdot\mathbf{E}_0 = 0, \text{ and } k = \frac{\omega_0}{c}\sqrt{\varepsilon_\mathrm{r}\mu_\mathrm{r}},$$

where c is the speed of light $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$. The latter equation is called the dispersion relation and it is an important equation. (3 points)

(b) Show that the magnetic field $\mathbf{H}(\mathbf{r},t)$ associated with the plane wave $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{-i(\omega_0 t - \mathbf{k}\cdot\mathbf{r})}$ can be written as a plane wave that has an amplitude of

$$\mathbf{H}_0 = \frac{\mathbf{u}\times\mathbf{E}_0}{Z},$$

where $Z = \sqrt{\frac{\mu_0\mu_\mathrm{r}}{\varepsilon_0\varepsilon_\mathrm{r}}}$ is the wave impedance of the medium. (2 points)

(c) Show that the time-averaged Poynting vector associated with the monochromatic plane wave discussed above can be written as

$$\langle \mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2}\Re\left[Z\right]\frac{\mid \mathbf{E}_0 \mid^2}{\mid Z \mid^2} e^{-k''\mathbf{u}\cdot\mathbf{r}}\mathbf{u},$$

where $k'' = \Im[k]$ and $\langle .. \rangle$ denotes a time-averaged quantity. (2 points)
(*Hint:* The Poynting vector is defined as $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)$ in real representation.)

(d) Show that in an inhomogeneous, nondispersive, and nonmagnetic dielectric medium (i.e. $\varepsilon_r(\mathbf{r},\omega) = \varepsilon_r(\mathbf{r})$ and $\mu(\mathbf{r},\omega) = \mu_0$), the electric and magnetic fields expressed in temporal Fourier (also called frequency) space satisfy the following wave equations:

$$\left(\nabla^2 + k_0^2\varepsilon_r(\mathbf{r})\right)\bar{\mathbf{E}}(\mathbf{r},\omega) = -\nabla(\bar{\mathbf{E}}(\mathbf{r},\omega)\cdot\nabla\ln\varepsilon_r(\mathbf{r})),$$
$$\left(\nabla^2 + k_0^2\varepsilon_r(\mathbf{r})\right)\bar{\mathbf{H}}(\mathbf{r},\omega) = (\nabla\times\bar{\mathbf{H}}(\mathbf{r},\omega))\times\nabla\ln\varepsilon_r(\mathbf{r}),$$

where $k_0 = \omega/c$. (3 points)
(*Hint:* $\frac{\nabla\Phi(\mathbf{r})}{\Phi(\mathbf{r})} = \nabla\ln\Phi(\mathbf{r})$, for a scalar function $\Phi(\mathbf{r})$.)

Problem 2. (5 points) Lorentz model of material dispersion

The electric susceptibility for a material with bound electrons that can interact resonantly with light at a specific frequency is given by a Lorentz model:

$$\chi(\omega) = \frac{\varepsilon_0 f}{(\omega^2 - \omega_0^2) - i\gamma\omega};$$

where f is the oscillator strength, γ the damping constant, and ω_0 the resonance frequency.

Calculate the response function

$$R(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \chi(\omega)e^{-i\omega t}\mathrm{d}\omega.$$

Discuss both cases $t < 0$ and $t > 0$. (5 points)

Problem 3. (9 points) Kramers-Kronig relation

(a) Consider a function $f(z)$ in the complex plane. $f(z)$ has a pole of order k at $z = z_0$. This means that $f(z)$ can be expanded about $z = z_0$ in the following Laurent series: $f(z) = \sum_{n=-k}^{+\infty} a_n(z - z_0)^n$. Show that for z_0 being a simple pole of order k=1 we have that:

$$\int_{C_{z_0}} f(z)\mathrm{d}z = i\pi \mathrm{Res}\left\{f(z = z_0)\right\}$$

where $\mathrm{Res}\left\{f(z = z_0)\right\} = a_{-1}$ is the residue of $f(z)$ at $z = z_0$ and C_{z_0} is the contour of half a circle centered at $z = z_0$, given by $z = z_0 + \lim_{R\to 0} Re^{i\theta}$, with $\theta \in [\pi, 2\pi]$. (3 points)

(b) Given that the imaginary part of the permittivity of a medium is

$$\Im[\varepsilon(\omega)] = \frac{\gamma\omega_p^2}{\omega(\gamma^2 + \omega^2)},$$

find the real part of the permittivity $\Re[\varepsilon(\omega)]$ by using the Kramers-Kronig relation.

(*Hint*: Use the formula $\Re[\varepsilon(\omega)] = 1 + \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{+\infty} \frac{\Im[\varepsilon(\bar{\omega})]}{\bar{\omega} - \omega}\mathrm{d}\bar{\omega}$, where 'PV' denotes the Cauchy's principal value.) (6 points)