Theoretical Optics

Exercise 3

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Drop point: Your tutorial group in ILIAS

Due Date: June 2nd 2022, 16:00

Problem 1. (15 points) Electrostatic stress tensor

Consider two spherical shells with a radius of R are charged with a surface charge density of σ, and are separated by a center-to-center distance $d > 2R$ in free space (see Figure below).

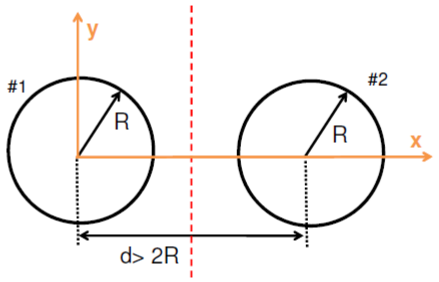

Figure 1: Two identical spherical shells in free space with a surface charge density of σ are depicted in x-y plane.

(a) Show that the total electric field caused by two spherical shells at the planar surface sketched with a red dashed-line ($x = \frac{d}{2}$) in Figure 1 is written as

$$\mathbf{E} = \frac{2\sigma R^2}{\epsilon_0}\left(\frac{1}{d^2/4 + y^2 + z^2}\right)^{3/2}(y\hat{y} + z\hat{z}).$$

(5 points)

(b) Calculate Maxwell's stress tensor $\overleftrightarrow{\mathbf{T}}$ at $x = d/2$. (2 points)

(c) By using the stress tensor $\overleftrightarrow{\mathbf{T}}$ obtained above, show that the force that the shell #1 exerts on the shell #2 is written as

$$\mathbf{F} = \frac{4\pi\sigma^2 R^4}{\epsilon_0 d^2}\hat{\mathbf{x}}.$$

(*Hint*: A proper surface for the surface integration is y-z plane at $x = d/2$.) (6 points)

(d) Verify that the calculated force above is identical to the force between two charged *point* particles when the charge of each particle is $Q = 4\pi R^2\sigma$. (2 points)

Problem 2. (9 points) Scalar diffraction theory

Consider the circularly symmetric object shown in Fig. 2. It is infinite in extent in the x-y plane. Its amplitude transmission function is given by

$$t_{\mathrm{A}}(r) = 2\pi J_0(ar) + 4\pi J_0(2ar),$$

where $J_0(x)$ is the zeroth-order Bessel function of the first kind, a is some positive real number signifying a spatial frequency (*i.e.* it has the units of m^{-1}), and $r = \sqrt{x^2 + y^2}$ is the radial coordinate in the two-dimensional plane. In scalar approximation, this object is illuminated by a normally incident, unit-amplitude plane wave propagating along z direction, and the paraxial condition is assumed to hold.

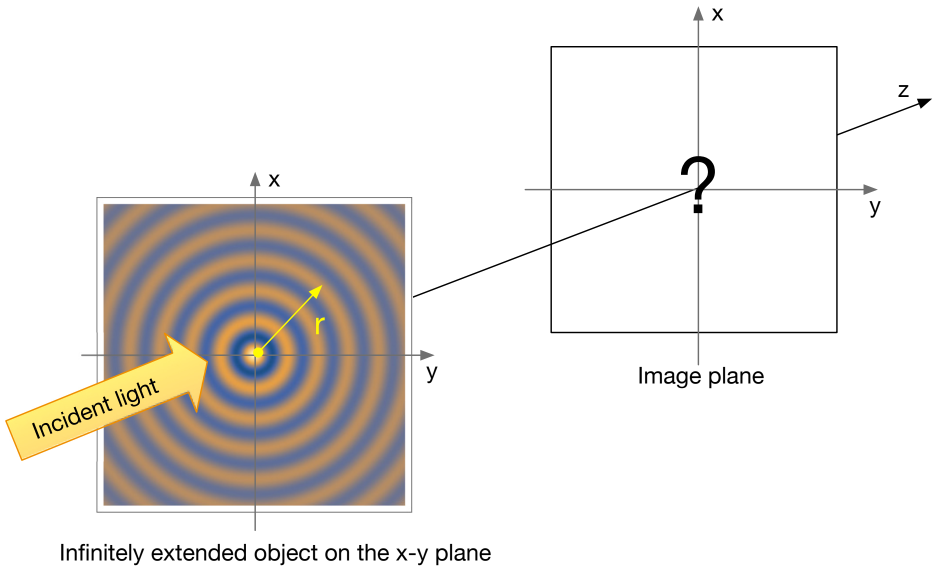

Figure 2: A circularly symmetric object in $x - y$ plane.

(a) Using Fresnel approximation, find the expression of the field distribution in the image plane at z. (4 points)

(*Hint*: Use the formula for the Fourier transformation ($\mathcal{FT}$) of the Bessel function:
$\frac{1}{(2\pi)^2}\iint 2\pi\gamma J_0(\gamma\sqrt{x^2+y^2})e^{-i(\alpha x+\beta y)}\mathrm{d}x\mathrm{d}y = \delta(\sqrt{\alpha^2+\beta^2}-\gamma)$, where γ is a some positive real constant.
Also use the property of the Dirac delta function:
$\mathcal{FT}[\delta(x-x_0)f(x)] = \mathcal{FT}[\delta(x-x_0)f(x_0)]$ where $f(x)$ has no singularity in the whole space.)

(b) Discuss the change of the field's amplitude in the direction of propagation z. (2 points)

(c) At what distances z behind the object, will we find a field distribution that is of the same form as that of the object, up to possible complex constants (*i.e.* ignore the overall phase factor)? (3 points)