Theoretical Optics

Exercise 4

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Drop point: Your tutorial group in ILIAS

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Problem 1. (7 points) Diffraction theory

In paraxial (Fresnel) approximation, any periodic field distribution $u_0(x+d) = u_0(x)$ is reproduced (except for a global phase factor) after propagation lengths z_m that are integer multiples of a distinct length L_T, called *Talbot length*. To prove that, we need to develop the following steps:

(a) Write the exact expression of the transfer function for propagation in homogeneous space. Furthermore, derive the Fresnel transfer function in homogeneous space. (1 point)
(*Hint:* Use the Taylor expansion $\sqrt{1-x} \approx 1 - \frac{1}{2}x$ for $x \ll 1$.)

(b) For an arbitrary field $u_0(x, z=0)$ expanded into a Fourier series, i.e., $u_0(x, z=0) = \sum_n a_n e^{\left(in\frac{2\pi}{d}x\right)}$, calculate the spatial Fourier spectrum $U_0(\alpha, z=0)$.
(*Useful formula:* $\frac{1}{2\pi}\int e^{i\beta x}e^{-i\alpha x}\mathrm{d}x = \delta(\beta - \alpha)$.) (2 points)

(c) Calculate the field $u(x,z)$ by means of the Fresnel transfer function in homogeneous space. (2 points)

(d) Show that the propagation lengths where the field reproduces (up to a global phase factor) are given by

$$z_m = mL_\mathrm{T},$$

where $L_\mathrm{T} = \frac{2d^2}{\lambda_0}$, and $m \in \mathbb{N}$. (2 points)

Problem 2. (10 points) Poisson's spot

Consider a plane wave that propagates in the positive z-direction and which impinges at normal incidence on a circular disk of radius a in free space, as shown in Figure 1. The scalar incident field can be written as $u(\mathbf{r}) = Ae^{ikz}$, where A is a real constant in x'-y' plane, *i.e.* a unit-amplitude illumination.

(a) Using the first Rayleigh Sommerfield diffraction formula, show that the field amplitude on the optical axis at the distance z away the center of the disk is given by

$$u_{RS}(0,0,z) = A\frac{z}{r_0}e^{ikr_0},$$

where $r_0^2 = a^2 + z^2$.

(*Hint*: Use the Sommerfeld lemma $\int_a^b f(x)e^{ikx}\mathrm{d}x \approx \frac{f(x)}{ik}e^{ikx}\Big|_a^b$, where the approximation is justified if $f(x)$ is a slowly varying function, or, equivalently, in the limit $k \to \infty$ ($\lambda \to 0$).) (2 points)

(b) By using the Helmholtz and Kirchhoff theorem, prove that for the plane wave incident field, the following Fresnel-Kirchhoff diffraction formula holds

$$u_{FK}(\mathbf{r}) = \frac{1}{2i\lambda}A\iint \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\left(1+\cos(\alpha)\right)\mathrm{d}^2r',$$

where α is the angle between the outward normal direction of the disk and the vector $\mathbf{r}' - \mathbf{r}$.

(Hint: Use $G(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}'-\mathbf{r}|}$ for Green's function, and the approximation $\frac{1}{|\mathbf{r}'-\mathbf{r}|^2} \approx 0$.) (4 points)

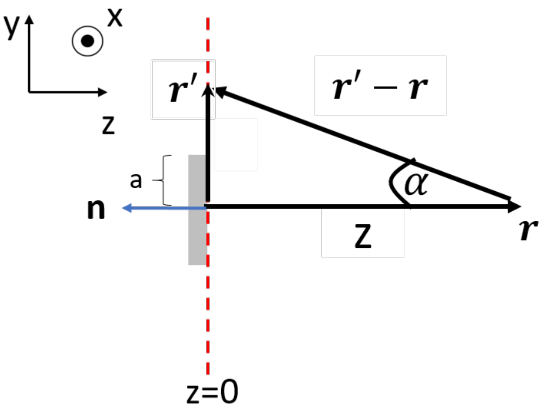

Figure 1: The diffraction problem of Poisson's spot. The main problem consists in the calculation of the field in the positive half-space ($z > 0$) for a circular mask in the x'-y' plane. Please note, the integration in the diffraction integrals goes over the entire x'-y' plane at $z = 0$. Eventually, we are interested here only for the fields at $x = y = 0$, *i.e.*, along the z-axis. The gray box denotes the area of the circular opaque disk that causes Poisson's spot at distance z. The arrow $\mathbf{r}'$ denotes the spatial coordinate of the plane just behind the disk, while the arrow $\mathbf{r}$ represents the point of our observation.

(c) Show that using Fresnel-Kirchoff diffraction formula, the field amplitude at the distance z from the center of the disk can be written as

$$u_{FK}(0,0,z) = \frac{A}{2} e^{ikr_0}\left(1 + \frac{z}{r_0}\right).$$

(*Hint*: Use the approximation $\int_{r_0}^{\infty} e^{ikr}\mathrm{d}r \approx -\frac{e^{ikr_0}}{ik}$ and the Sommerfeld lemma $\int_a^b f(x)e^{ikx}\mathrm{d}x \approx \frac{f(x)}{ik}e^{ikx}\Big|_a^b$.) (4 points)

Problem 3. (7 points) Fraunhofer approximation

Compute the diffraction pattern in Fraunhofer approximation for:

(a) A pinhole with radius a
$\left(Hint: \int_0^{2\pi} e^{-ix\cos\varphi}\mathrm{d}\varphi = 2\pi J_0(x) \text{ and } \int_0^a J_0(k\rho)\rho\mathrm{d}\rho = a^2\frac{J_1(ka)}{ka}\right)$ (4 points)

(b) A sequence of N pinholes placed along the x-axis with distances of $d > 2a$.
$\left(Hint: \sum_{n=0}^{\mathrm{N}-1} x^n = \frac{1-x^{\mathrm{N}}}{1-x}\right)$ (3 points)