Theoretical Optics

Exercise 1

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Drop point: Your tutorial group in ILIAS

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Problem 1. (10 points) Maxwell's equations in homogeneous and inhomogeneous media

Consider Maxwell's equations in a homogeneous, isotropic, dispersive, source-free, dielectric and magnetic medium characterized by a relative permittivity $\varepsilon_\mathrm{r}(\omega)$ and relative permeability $\mu_\mathrm{r}(\omega)$:

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \qquad \nabla \cdot \mathbf{B}(\mathbf{r},t) = 0,$$
$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}, \qquad \nabla \cdot \mathbf{D}(\mathbf{r},t) = 0.$$

(a) Show that if the electric field is a time harmonic plane wave characterized by a wave vector $\mathbf{k} = k\mathbf{u}$ and frequency ω, i.e. $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{-i(\omega_0 t - \mathbf{k}\cdot\mathbf{r})}$, then $\mathbf{k}$ satisfies the relations written as

$$\mathbf{k}\cdot\mathbf{E}_0 = 0, \text{ and } k = \frac{\omega_0}{c}\sqrt{\varepsilon_\mathrm{r}\mu_\mathrm{r}},$$

where c is the speed of light $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$. The latter equation is called the dispersion relation and it is an important equation. (3 points)

(b) Show that the magnetic field $\mathbf{H}(\mathbf{r},t)$ associated with the plane wave $\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{-i(\omega_0 t - \mathbf{k}\cdot\mathbf{r})}$ can be written as a plane wave that has an amplitude of

$$\mathbf{H}_0 = \frac{\mathbf{u}\times\mathbf{E}_0}{Z},$$

where $Z = \sqrt{\frac{\mu_0\mu_\mathrm{r}}{\varepsilon_0\varepsilon_\mathrm{r}}}$ is the wave impedance of the medium. (2 points)

(c) Show that the time-averaged Poynting vector associated with the monochromatic plane wave discussed above can be written as

$$\langle \mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2}\Re\left[Z\right]\frac{\mid \mathbf{E}_0 \mid^2}{\mid Z \mid^2} e^{-2k''\mathbf{u}\cdot\mathbf{r}}\mathbf{u},$$

where $k'' = \Im[k]$ and $\langle .. \rangle$ denotes a time-averaged quantity. (2 points)
(*Hint:* The Poynting vector is defined as $\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)$ in real representation.)

(d) Show that in an inhomogeneous, nondispersive, and nonmagnetic dielectric medium (i.e. $\varepsilon_r(\mathbf{r},\omega) = \varepsilon_r(\mathbf{r})$ and $\mu(\mathbf{r},\omega) = \mu_0$), the electric and magnetic fields expressed in temporal Fourier (also called frequency) space satisfy the following wave equations:

$$\left(\nabla^2 + k_0^2\varepsilon_r(\mathbf{r})\right)\bar{\mathbf{E}}(\mathbf{r},\omega) = -\nabla(\bar{\mathbf{E}}(\mathbf{r},\omega)\cdot\nabla\ln\varepsilon_r(\mathbf{r})),$$
$$\left(\nabla^2 + k_0^2\varepsilon_r(\mathbf{r})\right)\bar{\mathbf{H}}(\mathbf{r},\omega) = (\nabla\times\bar{\mathbf{H}}(\mathbf{r},\omega))\times\nabla\ln\varepsilon_r(\mathbf{r}),$$

where $k_0 = \omega/c$. (3 points)

(*Hint:* $\frac{\nabla\Phi(\mathbf{r})}{\Phi(\mathbf{r})} = \nabla\ln\Phi(\mathbf{r})$, for a scalar function $\Phi(\mathbf{r})$.)

Solution to problem 1

(a) To simplify the expression, we first make the transition to Frequency Fourier space of Maxwell's equations using the Ansatz $\mathbf{E}(\mathbf{r},t) = \bar{\mathbf{E}}(\mathbf{r},\omega)e^{-i\omega t}$. This allows to recast Maxwell's equations as an algebraic

$$\nabla \times \bar{\mathbf{E}}(\mathbf{r},\omega) = i\omega\bar{\mathbf{B}}(\mathbf{r},\omega), \qquad \nabla \cdot \bar{\mathbf{B}}(\mathbf{r},\omega) = 0,$$
$$\nabla \times \bar{\mathbf{H}}(\mathbf{r},\omega) = -i\omega\bar{\mathbf{D}}(\mathbf{r},\omega), \qquad \nabla \cdot \bar{\mathbf{D}}(\mathbf{r},\omega) = 0.$$

The equation $\nabla \cdot \bar{\mathbf{D}}(\mathbf{r},\omega) = 0$ can be rewritten in terms of the electric field, $\varepsilon_0\varepsilon_\mathrm{r}(\omega)\nabla \cdot \bar{\mathbf{E}}(\mathbf{r},\omega) = 0$. This equation can only be zero (excluding frequencies where the permittivity is zero) whenever $\nabla \cdot \bar{\mathbf{E}}(\mathbf{r},\omega) = 0$.

Now we furthermore make the Ansatz that the field corresponds to a plane wave with a spatially dependent field distribution that corresponds to $\bar{\mathbf{E}}(\mathbf{r},\omega) = \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{r}}$. If we substitute this ansatz we get

$$i(k_x E_{0_x} + k_y E_{0_y} + k_z E_{0_z})e^{i\mathbf{k}\cdot\mathbf{r}} = 0,$$

This requires that $\mathbf{k}\cdot\mathbf{E}_0 = 0$.

To derive the dispersion relation we apply the curl equation one more time to the first Maxwell's equations

$$\nabla \times \nabla \times \bar{\mathbf{E}}(\mathbf{r},\omega) = i\omega\nabla \times \bar{\mathbf{B}}(\mathbf{r},\omega)$$

Plugging in the second curl equation and using the fact that $\bar{\mathbf{B}}(\mathbf{r},\omega) = \mu_0\mu_r(\omega)\bar{\mathbf{H}}(\mathbf{r},\omega)$ we obtain

$$\nabla \times \nabla \times \bar{\mathbf{E}}(\mathbf{r},\omega) = \omega^2\varepsilon_0\mu_0\varepsilon_r(\omega)\mu_r(\omega)\bar{\mathbf{E}}(\mathbf{r},\omega)$$

Now using the fact that $\nabla\times\nabla\times = \nabla[\nabla\cdot] - \nabla^2$ we can conclude that

$$\nabla \times \nabla \times \bar{\mathbf{E}}(\mathbf{r},\omega) = \nabla[\nabla\cdot\bar{\mathbf{E}}(\mathbf{r},\omega)] - \nabla^2\bar{\mathbf{E}}(\mathbf{r},\omega) = -\nabla^2\bar{\mathbf{E}}(\mathbf{r},\omega)$$

The resulting Helmholtz equation reads as

$$\nabla^2\bar{\mathbf{E}}(\mathbf{r},\omega) + \frac{\omega^2}{c_0^2}\varepsilon_r(\omega)\mu_r(\omega)\bar{\mathbf{E}}(\mathbf{r},\omega) = 0$$

with $\nabla^2\bar{\mathbf{E}}(\mathbf{r},\omega) = \nabla(\nabla\cdot\bar{\mathbf{E}}(\mathbf{r},\omega)) = -\mathbf{k}^2\bar{\mathbf{E}}(\mathbf{r},\omega)$ we obtain

$$\left(-\mathbf{k}^2 + \frac{\omega^2}{c_0^2}\varepsilon_r(\omega)\mu_r(\omega)\right)\bar{\mathbf{E}}(\mathbf{r},\omega) = 0$$

which is only possible to be solved if

$$\mathbf{k}^2 = k_x^2 + k_y^2 + k_z^3 = k^2 = \frac{\omega^2}{c_0^2}\varepsilon_r(\omega)\mu_r(\omega).$$

(b) This can be derived straightforwardly from the Faraday-Neumann equation, if we substitute the plane wave solution

$$i\mathbf{k}\times\mathbf{E}_0 e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})} = -\mu_0\mu_\mathrm{r}(-i\omega)\mathbf{H}_0 e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})},$$

that leads to

$$\mathbf{H}_0 = \frac{k}{\omega\mu_0\mu_\mathrm{r}}\mathbf{u}\times\mathbf{E}_0.$$

The expression $\frac{k}{\omega\mu_0\mu_\mathrm{r}}$ can be written as

$$\frac{k}{\omega\mu_0\mu_\mathrm{r}} = \frac{\omega\sqrt{\mu_0\varepsilon_0\varepsilon_\mathrm{r}\mu_\mathrm{r}}}{\omega\mu_0\mu_\mathrm{r}} = \sqrt{\frac{\varepsilon_0\varepsilon_\mathrm{r}}{\mu_0\mu_\mathrm{r}}} = \frac{1}{Z},$$

so that we can conclude

$$\mathbf{H}_0 = \frac{\mathbf{u}\times\mathbf{E}_0}{Z}.$$

(c) The Poynting vector in real representation reads

$$\begin{aligned}\mathbf{S}(\mathbf{r},t) &= \Re\{\mathbf{E}(\mathbf{r},t)\} \times \Re\{\mathbf{H}(\mathbf{r},t)\} \\ &= \frac{\mathbf{E}(\mathbf{r},t) + \mathbf{E}^*(\mathbf{r},t)}{2} \times \frac{\mathbf{H}(\mathbf{r},t) + \mathbf{H}^*(\mathbf{r},t)}{2} \\ &= \frac{\bar{\mathbf{E}}(\mathbf{r},t)e^{-i\omega t} + \bar{\mathbf{E}}^*(\mathbf{r},t)e^{i\omega t}}{2} \times \frac{\bar{\mathbf{H}}(\mathbf{r},t)e^{-i\omega t} + \bar{\mathbf{H}}^*(\mathbf{r},t)e^{i\omega t}}{2} \\ &= \frac{1}{4}\left(\bar{\mathbf{E}} \times \bar{\mathbf{H}}^* + \bar{\mathbf{E}}^* \times \bar{\mathbf{H}} + \bar{\mathbf{E}} \times \bar{\mathbf{H}}e^{-2\omega t} + \bar{\mathbf{E}}^* \times \bar{\mathbf{H}}_0^* e^{2i\omega t}\right) \\ &= \frac{1}{2}\left(\Re\{\bar{\mathbf{E}} \times \bar{\mathbf{H}}^*\} + \Re\{\bar{\mathbf{E}} \times \bar{\mathbf{H}}e^{-2\omega t}\}\right)\end{aligned}$$

with $\bar{\mathbf{E}} = \mathbf{E}_0 e^{ik\mathbf{u}\cdot\mathbf{r}}$ and $\bar{\mathbf{H}} = \mathbf{H}_0 e^{ik\mathbf{u}\cdot\mathbf{r}}$.
The time average of the Poynting vector corresponds to the integral of the Poynting vector over time (duration of the measurement). The integration causes vanishing of the fastly oscillating terms so the time averaged Poyinting vector can be calculated as

$$\begin{aligned}\langle\mathbf{S}(\mathbf{r},t)\rangle &= \frac{1}{2}\Re\left[\bar{\mathbf{E}}(\mathbf{r},\omega_0) \times \bar{\mathbf{H}}^*(\mathbf{r},\omega_0)\right] = \frac{1}{2}\Re\left[\mathbf{E}_0 e^{ik\mathbf{u}\cdot\mathbf{r}} \times \mathbf{H}_0^* e^{-ik^*\mathbf{u}\cdot\mathbf{r}}\right] = \\ &= \frac{1}{2}\Re\left[e^{i(k-k^*\mathbf{u}\cdot\mathbf{r})}\mathbf{E}_0 \times \frac{\mathbf{u} \times \mathbf{E}_0^*}{Z^*}\right] = \frac{1}{2}e^{-2k''\mathbf{u}\cdot\mathbf{r}}\Re\left[\frac{(\mathbf{E}_0 \cdot \mathbf{E}_0^*)\mathbf{u} - (\mathbf{E}_0 \cdot \mathbf{u})\mathbf{E}_0}{Z^*}\right] = \\ &= \frac{1}{2}e^{-2k''\mathbf{u}\cdot\mathbf{r}}\Re\left[Z\right]\frac{\mid \mathbf{E}_0 \mid^2}{\mid Z \mid^2}\mathbf{u}\end{aligned}$$

(d) Because in this instance the medium is inhomogeneous, the fourth Maxwell equation reads

$$\nabla \cdot \bar{\mathbf{D}}(\mathbf{r},\omega) = \nabla \cdot \left(\varepsilon_0\varepsilon_r(\mathbf{r})\bar{\mathbf{E}}(\mathbf{r},\omega)\right) = \varepsilon_0\nabla \cdot \left(\varepsilon_r(\mathbf{r})\bar{\mathbf{E}}(\mathbf{r},\omega)\right) = 0,$$

since $n = \sqrt{\varepsilon_r}$. This implies,

$$\begin{aligned}\nabla(\varepsilon_r(\mathbf{r})) \cdot \bar{\mathbf{E}}(\mathbf{r},\omega) + \varepsilon_r(\mathbf{r})\nabla \cdot \bar{\mathbf{E}}(\mathbf{r},\omega) &= 0 \\ \Rightarrow \nabla \cdot \bar{\mathbf{E}}(\mathbf{r},\omega) = -\frac{\bar{\mathbf{E}}(\mathbf{r},\omega) \cdot \nabla(\varepsilon_r(\mathbf{r}))}{\varepsilon_r(\mathbf{r})} &= -\bar{\mathbf{E}} \cdot \nabla \ln \varepsilon_r(\mathbf{r}). \qquad (1)\end{aligned}$$

Now, the first and second Maxwell equations can be written as

$$\nabla \times \bar{\mathbf{E}}(\mathbf{r},\omega) = i\omega\mu_0\bar{\mathbf{H}}(\mathbf{r},\omega), \qquad \nabla \times \bar{\mathbf{H}}(\mathbf{r},\omega) = -i\omega\varepsilon_0\varepsilon_r(\mathbf{r})\bar{\mathbf{E}}(\mathbf{r},\omega).$$

By applying the curl operator $\nabla\times$ to the first Maxwell equation, and replacing $\nabla\times\bar{\mathbf{H}}(\mathbf{r},\omega)$ by its expression in terms of $\bar{\mathbf{E}}(\mathbf{r},\omega)$, we obtain

$$\nabla \times \nabla \times \bar{\mathbf{E}}(\mathbf{r},\omega) = \nabla[\nabla \cdot \bar{\mathbf{E}}(\mathbf{r},\omega)] - \nabla^2\bar{\mathbf{E}}(\mathbf{r},\omega) = \omega^2\mu_0\varepsilon_0\varepsilon_r(\mathbf{r})\bar{\mathbf{E}}(\mathbf{r},\omega).$$

Inserting equation (1) into the above result, it follows

$$\left(\nabla^2 + k_0^2\varepsilon_r(\mathbf{r})\right)\bar{\mathbf{E}}(\mathbf{r},\omega) = -\nabla(\bar{\mathbf{E}}(\mathbf{r},\omega) \cdot \nabla \ln \varepsilon_r(\mathbf{r})).$$

We proceed analogously for the magnetic field, using the fact that $\nabla \cdot \bar{\mathbf{H}}(\mathbf{r},\omega) = 0$, from the third Maxwell equation. We have

$$-\nabla^2\bar{\mathbf{H}}(\mathbf{r},\omega) = -i\omega\mu_0\varepsilon_0\nabla \times (\varepsilon_r(\mathbf{r})\bar{\mathbf{E}}(\mathbf{r},\omega)).$$

Now,

$$\begin{aligned}\nabla \times (\varepsilon_r(\mathbf{r})\bar{\mathbf{E}}(\mathbf{r},\omega)) &= \varepsilon_r(\mathbf{r})\nabla \times \bar{\mathbf{E}}(\mathbf{r},\omega) + \nabla\varepsilon_r(\mathbf{r}) \times \bar{\mathbf{E}}(\mathbf{r},\omega) \\ &= i\omega\mu_0\varepsilon_r(\mathbf{r})\bar{\mathbf{H}}(\mathbf{r},\omega) + i\frac{\nabla\varepsilon_r(\mathbf{r}) \times \nabla \times \bar{\mathbf{H}}(\mathbf{r},\omega)}{\varepsilon_r(\mathbf{r})\omega\varepsilon_0}\end{aligned}$$

Therefore,

$$-\nabla^2\bar{\mathbf{H}}(\mathbf{r},\omega) = -i\omega\varepsilon_0\left(i\omega\mu_0\varepsilon_r(\mathbf{r})\bar{\mathbf{H}}(\mathbf{r},\omega) + i\frac{\nabla \ln \varepsilon_r(\mathbf{r}) \times \nabla \times \bar{\mathbf{H}}(\mathbf{r},\omega)}{\omega\varepsilon_0}\right)$$

Simplifying and rearranging, we finally obtain

$$\left(\nabla^2 + k_0^2\varepsilon_r(\mathbf{r})\right)\bar{\mathbf{H}}(\mathbf{r},\omega) = (\nabla \times \bar{\mathbf{H}}(\mathbf{r},\omega)) \times \nabla \ln \varepsilon_r(\mathbf{r}),$$

Problem 2. (5 points) Lorentz model of material dispersion

The electric susceptibility for a material with bound electrons that can interact resonantly with light at a specific frequency is given by a Lorentz model:

$$\chi(\omega) = \frac{\varepsilon_0 f}{(\omega_0^2 - \omega^2) - i\gamma\omega};$$

where f is the oscillator strength, γ the damping constant, and ω_0 the resonance frequency.

Calculate the response function

$$R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \chi(\omega) e^{-i\omega t} \mathrm{d}\omega.$$

Discuss both cases $t < 0$ and $t > 0$. (5 points)

Solution to problem 2

The response function can be written as:

$$R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varepsilon_0 f}{(\omega_0^2 - \omega^2) - i\gamma\omega} e^{-i\omega t} d\omega.$$

The integral can be evaluated by means of the residue theory. The poles of the function $\chi(\omega)$ are given by:

$$\omega_{1,2} = -\frac{i\gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} = -\frac{i\gamma}{2} \pm \Omega,$$

with $\omega_0^2 \gg \gamma^2$. In this case there are only two poles of the first order in the negative complex half-space. We can now perform an integration along a closed half-circle in the complex plane.

(i) Case t<0. In this case we must use a half-circle in the upper half-plane, since the integrand $\longrightarrow 0$ along the half-circle. In this case we get:

$$R(t) = \int_{-\infty}^{\infty} \frac{\varepsilon_0 f}{\left(\omega + \frac{i\gamma}{2} - \Omega\right)\left(\omega + \frac{i\gamma}{2} + \Omega\right)} e^{-i\omega t} \mathrm{d}\omega = 0$$

since there are no residues in the half-circle.

(ii) Case t>0. In this case we must use a half-circle in the lower half-plane, since the integrand $\longrightarrow 0$ along the half-circle. We can then write:

$$R(t) = \int_{-\infty}^{\infty} \frac{\varepsilon_0 f}{\left(\omega + \frac{i\gamma}{2} - \Omega\right)\left(\omega + \frac{i\gamma}{2} + \Omega\right)} e^{-i\omega t} \mathrm{d}\omega = i\left(\mathrm{Res}\left(\chi, \frac{-i\gamma}{2} + \Omega\right) + \mathrm{Res}\left(\chi, \frac{-i\gamma}{2} - \Omega\right)\right).$$

This can be evaluated as:

$$\begin{aligned}
&= i\left[\left(\omega + \frac{i\gamma}{2} - \Omega\right) \frac{\varepsilon_0 f}{\left(\omega + \frac{i\gamma}{2} - \Omega\right)\left(\omega + \frac{i\gamma}{2} + \Omega\right)} e^{-i\omega t}\right]\Bigg|_{\omega = -\frac{i\gamma}{2} + \Omega} + \\
&\quad + i\left[\left(\omega + \frac{i\gamma}{2} + \Omega\right) \frac{\varepsilon_0 f}{\left(\omega + \frac{i\gamma}{2} - \Omega\right)\left(\omega + \frac{i\gamma}{2} + \Omega\right)} e^{-i\omega t}\right]\Bigg|_{\omega = -\frac{i\gamma}{2} - \Omega} = \\
&= i\frac{\varepsilon_0 f}{2\Omega}\left(e^{-i\Omega t} + e^{i\Omega t}\right) e^{-\frac{\gamma t}{2}} = \frac{\varepsilon_0 f}{\Omega} \sin(\Omega t)\, e^{-\frac{\gamma t}{2}}
\end{aligned}$$

Problem 3. (9 points) Kramers-Kronig relation

(a) Consider a function $f(z)$ in the complex plane. $f(z)$ has a pole of order k at $z = z_0$. This means that $f(z)$ can be expanded about $z = z_0$ in the following Laurent series: $f(z) = \sum_{n=-k}^{+\infty} a_n (z - z_0)^n$. Show that for z_0 being a simple pole of order k=1 we have that:

$$\int_{C_{z_0}} f(z)\mathrm{d}z = i\pi \mathrm{Res}\left\{f(z = z_0)\right\}$$

where $\mathrm{Res}\left\{f(z = z_0)\right\} = a_{-1}$ is the residue of $f(z)$ at $z = z_0$ and C_{z_0} is the contour of half a circle centered at $z = z_0$, given by $z = z_0 + \lim_{R\to 0} Re^{i\theta}$, with $\theta \in [\pi, 2\pi]$. (3 points)

(b) Given that the imaginary part of the permittivity of a medium is

$$\Im[\varepsilon(\omega)] = \frac{\gamma\omega_p^2}{\omega(\gamma^2 + \omega^2)},$$

find the real part of the permittivity $\Re[\varepsilon(\omega)]$ by using the Kramers-Kronig relation.

(*Hint*: Use the formula $\Re[\varepsilon(\omega)] = 1 + \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{+\infty} \frac{\Im[\varepsilon(\bar{\omega})]}{\bar{\omega}-\omega}\mathrm{d}\bar{\omega}$, where 'PV' denotes the Cauchy's principal value.) (6 points)

Solution to problem 3

(a) We have that:

$$\begin{aligned}
\int_{C_{z_0}} f(z)dz &= \int_{C_{z_0}} \sum_{n=-k}^{+\infty} a_n (z - z_0)^n \mathrm{d}z \\
&\overset{z=z_0+Re^{i\theta}}{=} \sum_{n=-k}^{+\infty} a_n \lim_{R\to 0} \int_{\pi}^{2\pi} (Re^{i\theta})^n iRe^{i\theta}\mathrm{d}\theta \\
&= i\sum_{n=-k}^{+\infty} a_n \lim_{R\to 0} R^{n+1} \int_{\pi}^{2\pi} e^{i(n+1)\theta}\mathrm{d}\theta \\
&= i\sum_{n=-k}^{+\infty} a_n \lim_{R\to 0} R^{n+1} \left[\pi\delta_{n,-1} + \frac{1+(-1)^n}{i(n+1)}\delta_{n,m\neq -1}\right] \\
&= i\pi a_{-1}, \qquad \text{for} \quad k = 1 \\
&= i\pi \mathrm{Res}\left\{f(z = z_0)\right\}, \qquad \text{for} \quad k = 1
\end{aligned}$$

(b) We have that:

$$\begin{aligned}
\Re[\varepsilon(\omega)] &= 1 + \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{\Im[\varepsilon(\overline{\omega})]}{\overline{\omega} - \omega}\mathrm{d}\overline{\omega} \\
&= 1 + \frac{\gamma\omega_p^2}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{\overline{\omega}(\gamma^2 + \overline{\omega}^2)(\overline{\omega} - \omega)}\mathrm{d}\overline{\omega}.
\end{aligned}$$

The integrated quantity has four poles of the first order at $\overline{\omega} = 0$, $\overline{\omega} = \omega$, and $\overline{\omega} = \pm i\gamma$. We proceed with the integration along a contour C that encloses the upper half complex plane. We can break contour C as the sum of the contours S_∞, S_0, S_ω, S. S_∞ is given by $z = \lim_{R\to\infty} Re^{i\theta}$, $\theta \in [0, \pi]$ S_0 is given by $z = \lim_{R\to 0} Re^{i\theta}$, $\theta \in [\pi, 2\pi]$. S_ω is given by $z = \omega + \lim_{R\to 0} Re^{i\theta}$, $\theta \in [\pi, 2\pi]$.
Finally, S is given by $z \in (-\infty, -\lim_{R\to 0} R] \cup [\lim_{R\to 0} R, \omega - \lim_{R\to 0} R] \cup [\omega + \lim_{R\to 0} R, +\infty)$. We want to calculate the following integral:

$$
\begin{aligned}
I(\omega) &= \mathrm{PV}\int_{-\infty}^{\infty} f(\varpi)\mathrm{d}\varpi \\
&= \mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{\varpi(\gamma^2+\varpi^2)(\varpi-\omega)}\mathrm{d}\varpi \\
&= \int_S \frac{1}{\varpi(\gamma^2+\varpi^2)(\varpi-\omega)}\mathrm{d}\varpi \\
&= \left[\int_C - \int_{S_\infty} - \int_{S_0} - \int_{S_\omega}\right] \frac{1}{\varpi(\gamma^2+\varpi^2)(\varpi-\omega)}\mathrm{d}\varpi
\end{aligned}
$$

From the Residue Theorem, since contour C encloses three simple poles at $\varpi = i\gamma$, $\varpi = 0$, $\varpi = \omega$, we have that:

$$
\int_C \frac{1}{\varpi(\gamma^2+\varpi^2)(\varpi-\omega)}\mathrm{d}\varpi = 2\pi i\mathrm{Res}\left\{f(\varpi = i\gamma)\right\} + 2\pi i\mathrm{Res}\left\{f(\varpi = 0)\right\} + 2\pi i\mathrm{Res}\left\{f(\varpi = \omega)\right\}
$$

Also, since $|\varpi(\gamma^2+\varpi^2)(\varpi-\omega)| \to \infty$ as $|\varpi| \to \infty$, we have that:

$$
\int_{S_\infty} \frac{1}{\varpi(\gamma^2+\varpi^2)(\varpi-\omega)}\mathrm{d}\varpi = 0
$$

Moreover, from question (a) we have:

$$
\begin{aligned}
\int_{S_0} \frac{1}{\varpi(\gamma^2+\varpi^2)(\varpi-\omega)}\mathrm{d}\varpi &= \pi i\mathrm{Res}\left\{f(\varpi = 0)\right\} \\
\int_{S_\omega} \frac{1}{\varpi(\gamma^2+\varpi^2)(\varpi-\omega)}\mathrm{d}\varpi &= \pi i\mathrm{Res}\left\{f(\varpi = \omega)\right\}
\end{aligned}
$$

Putting all the above together gives:

$$
\begin{aligned}
I(\omega) &= 2\pi i\mathrm{Res}\left\{f(\varpi = i\gamma)\right\} + \pi i\mathrm{Res}\left\{f(\varpi = 0)\right\} + \pi i\mathrm{Res}\left\{f(\varpi = \omega)\right\} \\
&= 2\pi i \frac{1}{\varpi(i\gamma+\varpi)(\varpi-\omega)}\Big|_{\varpi=i\gamma} + \pi i \frac{1}{(\gamma^2+\varpi^2)(\varpi-\omega)}\Big|_{\varpi=0} + \pi i \frac{1}{\varpi(\gamma^2+\varpi^2)}\Big|_{\varpi=\omega} \\
&= \pi i\left(\frac{\omega+i\gamma}{\gamma^2(\omega^2+\gamma^2)} - \frac{1}{\gamma^2\omega} + \frac{1}{\omega(\omega^2+\gamma^2)}\right) \\
&= -\frac{\pi}{\gamma(\omega^2+\gamma^2)}.
\end{aligned}
$$

Finally. we get:

$$
\begin{aligned}
\Re[\varepsilon(\omega)] &= 1 + \frac{\gamma\omega_p^2}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{1}{\varpi(\gamma^2+\varpi^2)(\varpi-\omega)}\mathrm{d}\varpi \\
&= 1 - \frac{\gamma\omega_p^2}{\pi}\frac{\pi}{\gamma(\omega^2+\gamma^2)} \\
&= \frac{\omega^2+\gamma^2-\omega_p^2}{\omega^2+\gamma^2}.
\end{aligned}
$$