Theoretical Optics

Exercise 2

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___/24 points

Drop point: Your tutorial group in ILIAS

Due Date: May 19th 2022, 16:00

Problem 1. (13 points) Pulse propagation in dispersive media

A Gaussian pulse with spectrum $U(z=0,\omega) = \frac{Tu_0}{\sqrt{2\pi}} e^{-\frac{T^2(\omega-\omega_0)^2}{2}}$, where u_0 is the pulse amplitude, T is the pulse duration and ω_0 the carrier frequency, propagates through a piece of glass with a thickness of $\Delta z = d$ and a dielectric function $\varepsilon(\omega)$.

(a) Expand the dispersion relation $k(\omega)$ in the neighborhood of $\omega = \omega_0$ up to the quadratic term. (2 points)

(b) Discuss the physical meaning of the three constant coefficients that appear in the expansion. (3 points)

(c) Write down an expression of the spectrum at the output, $U(z=d,\omega)$. (2 points)

(d) Compute the pulse wave form at the output, $u(z=d,t)$. (4 points)

(e) Write down an expression of the reformed pulse duration at the output. (1 point)

(f) Discuss the dependency of the term $\frac{\mathrm{d}^2k}{\mathrm{d}\omega^2}\Big|_{\omega=\omega_0}$ and the distance d for the reformed pulse duration. (1 point)

(*Hint* 1: Use the answer for question 1(a) in the other questions.)
(*Hint* 2: Use the formula $\int_{-\infty}^{\infty} e^{-AX^2+BX}\mathrm{d}X = \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{4A}}$ for $\mathrm{Re}[A] > 0$.)

Solution to problem 1

(a) The dispersion relation can be expanded as:

$$k(\omega) = k(\omega_0) + \frac{\mathrm{d}k}{\mathrm{d}\omega}\Big|_{\omega=\omega_0} (\omega-\omega_0) + \frac{1}{2}\frac{\mathrm{d}^2k}{\mathrm{d}\omega^2}\Big|_{\omega=\omega_0} (\omega-\omega_0)^2 .$$

(b) The first term is $k(\omega_0) = \frac{\omega_0}{v_{ph}}$, where v_{ph} is the phase velocity.

The second term is $\frac{\mathrm{d}k}{\mathrm{d}\omega}\Big|_{\omega=\omega_0} = \frac{1}{v_g(\omega)}\Big|_{\omega=\omega_0}$, where v_g is the group velocity.

The last term is $\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{1}{v_g(\omega)}\Big|_{\omega=\omega_0} \equiv D$, which characterizes the frequency dependency of the group velocity, called the group velocity dispersion (GVD). If this term is zero, it means the group velocity does not change with frequency.

(c) The expression of the spectrum at the output is:

$$U(z=d,\omega) = \frac{Tu_0}{\sqrt{2\pi}} e^{-\frac{T^2(\omega-\omega_0)^2}{2}} e^{ik_0d + iv_g^{-1}(\omega-\omega_0)d + i\frac{D(\omega-\omega_0)^2 d}{2}} .$$

(d) The pulse waveform at the output is given by

$$u(z=d,t) = \int_{-\infty}^{\infty} U(z=d,\omega)e^{-i\omega t}\mathrm{d}\omega = \frac{Tu_0}{\sqrt{2\pi}} e^{i(k_0d-\omega_0t)} \int_{-\infty}^{\infty} e^{-\left(\frac{T^2}{2} - i\frac{Dd}{2}\right)(\omega-\omega_0)^2 + iv_g^{-1}(\omega-\omega_0)d - i(\omega-\omega_0)t}\mathrm{d}\omega.$$

The integral can be evaluated by means of the formula on the integration of a Gaussian, yielding:

$$\begin{aligned}
u(z=d,t) &= \frac{Tu_0}{\sqrt{2\pi}} e^{i(k_0 d-\omega_0 t)} \sqrt{\frac{\pi}{\left(\frac{T^2}{2}-i\frac{Dd}{2}\right)}}\, e^{\frac{-\left(t-\frac{d}{v_g}\right)^2}{4\left(\frac{T^2}{2}-i\frac{Dd}{2}\right)}}, \\
&= \frac{Tu_0}{\sqrt{2\pi}} e^{i(k_0 d-\omega_0 t)} \sqrt{\frac{\pi}{\frac{T^2}{2}\left(1-i\frac{Dd}{T^2}\right)}}\, e^{\frac{-\left(t-\frac{d}{v_g}\right)^2\left(1+i\frac{Dd}{T^2}\right)}{2T^2\left(1-i\frac{Dd}{T^2}\right)\left(1+i\frac{Dd}{T^2}\right)}}, \\
&= u_0 e^{i(k_0 d-\omega_0 t)} \sqrt{\frac{1}{(1-i\Phi)}}\, e^{\frac{-\left(t-\frac{d}{v_g}\right)^2(1+i\Phi)}{2T^2(1+\Phi^2)}}, \\
&= u_0 \sqrt{\frac{1}{(1-i\Phi)}}\, e^{\frac{-\left(t-\frac{d}{v_g}\right)^2}{2\tau^2}} e^{-i\frac{\left(t-\frac{d}{v_g}\right)^2\Phi}{2\tau^2}} e^{i(k_0 d-\omega_0 t)},
\end{aligned}$$

where $\Phi = \frac{Dd}{T^2}$, and $\tau = T\sqrt{1+\Phi^2}$.

(e) The pulse duration at the output is τ.

(f) The reformed pulse duration τ increases with the distance d and D.

Problem 2. (11 points) Poynting Vector and Normal Mode

Consider a monochromatic plane wave of frequency ω_0, propagating in a homogeneous isotropic weakly lossy dielectric medium of relative permittivity $\varepsilon = \varepsilon' + i\varepsilon''$ (where $\varepsilon', \varepsilon'' > 0$ and $\varepsilon' >> \varepsilon''$). Its electric field has the form $\mathbf{E}_r(\mathbf{r},t) = E_0\, \mathbf{e}_x\, e^{-\alpha z}\, \cos(\beta z - \omega_0 t + \phi)$, where the subscript r is used for the real valued fields.

(a) Even though the field is given here as a real valued quantity, in some of the following tasks, it might be better to use the complex representation as a mean to simplify your calculations. Therefore, write at first the same field as above but in complex notation. (1 point)

(b) By starting from the dispersion relation of the plane wave in the medium, show that

$$\beta \approx \frac{\omega_0}{c}\sqrt{\epsilon'} \quad \text{and} \quad \alpha \approx \frac{\omega_0}{c}\frac{\varepsilon''}{2\sqrt{\varepsilon'}}$$

Useful formula: $\sqrt{1+z} \approx 1 + \frac{1}{2}z, \quad z \in \mathbb{C}, \quad |z| \ll 1.$ (2 points)

(c) Start from Maxwell's equations to find the real valued magnetic field $\mathbf{H}_r(\mathbf{r},t)$. (4 points)

(d) Continue to use the real valued representation of the field to write down the formula for the instantaneous Poynting vector, $\mathbf{S}_r(\mathbf{r},t)$. (1 point)

(e) Find the time averaged Poynting vector using the formula $\langle \mathbf{S}_r(\mathbf{r},t)\rangle = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} \mathbf{S}_r(\mathbf{r},t)\mathrm{d}t$.
Useful formula: $\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]$, $\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]$. (3 points)

Solution to problem 2

(a) The real valued electric field can be written in its complex form as

$$\mathbf{E}_r(\mathbf{r},t) = E_0\, \mathbf{e}_x\, e^{-\alpha z}\, \cos(\beta z - \omega_0 t + \phi) = \frac{1}{2}\left[\mathbf{E}_c\, e^{-i\omega_0 t} + \mathbf{E}_c^*\, e^{+i\omega_0 t}\right]$$

with $\mathbf{E}_c = E_0\, e^{i\phi}\mathbf{e}_x\, e^{i(\beta+i\alpha)z}$.

(b) The wave-vector can be written as $\mathbf{k} = \mathbf{k}' + i\mathbf{k}'' = (\beta + i\alpha)\,\mathbf{e}_z$. We know the dispersion relation of a plane wave in a homogeneous medium as $\mathbf{k}\cdot\mathbf{k} = \frac{\omega_0^2}{c^2}\epsilon$. This leads to:

$$\beta + i\alpha = \frac{\omega_0}{c}\sqrt{\varepsilon' + i\varepsilon''} = \frac{\omega_0}{c}\sqrt{\varepsilon'}\sqrt{1 + i\frac{\varepsilon''}{\varepsilon'}},$$

The expression can be Taylor expanded, and we obtain

$$\beta + i\alpha = \frac{\omega_0}{c}\sqrt{\varepsilon_1}\left(1 + i\frac{\varepsilon''}{2\varepsilon'}\right)$$

From which we find $k' \approx \frac{\omega_0}{c}\sqrt{\epsilon'}$ and $k'' \approx \frac{\omega_0}{c}\frac{\varepsilon''}{2\sqrt{\varepsilon'}}$.

(c) Same like electric field, we can present the real valued magnetic field like:

$$\mathbf{H}_r(\mathbf{r},t) = \frac{1}{2}\left[\mathbf{H}_c\ e^{-i\omega_0 t} + \mathbf{H}_c^*\ e^{+i\omega_0 t}\right]$$

Using the time domain Maxwell equation $\nabla \times \mathbf{E}_r(\mathbf{r},t) = -\mu_0\frac{\partial \mathbf{H}_r(\mathbf{r},t)}{\partial t}$, we find the relation between the complex amplitudes to be $\nabla \times \mathbf{E}_c = i\omega\mu_0\mathbf{H}_c$. Followed by:

$$\nabla \times \mathbf{E}_c = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0\ e^{i\phi}\mathbf{e}_x\ e^{i(\beta+i\alpha)z} & 0 & 0 \end{vmatrix} = i(\beta+i\alpha)E_0\ e^{i\phi}\mathbf{e}_y\ e^{i(\beta+i\alpha)z}$$

Which gives $\mathbf{H}_c = \frac{(\beta+i\alpha)E_0\ e^{i\phi}}{\omega_0\mu_0}\mathbf{e}_y\ e^{i(\beta+i\alpha)z}$. And we calculate the real valued magnetic field:

$$\begin{aligned} \mathbf{H}_r(\mathbf{r},t) = \frac{E_0 e^{-\alpha z}}{2\omega\mu_0}\mathbf{e}_y\left[(\beta+i\alpha)e^{i(\beta z-\omega_0 t+\phi)} + (\beta-i\alpha)e^{-i(\beta z-\omega_0 t+\phi)}\right] = \\ = \frac{E_0}{\omega_0\mu_0}e^{-\alpha z}\mathbf{e}_y\left[\beta\cos(\beta z-\omega_0 t+\phi) - \alpha\sin(\beta z-\omega_0 t+\phi)\right] \end{aligned}$$

(d) $\mathbf{S}_r(\mathbf{r},t) = \mathbf{E}_r(\mathbf{r},t) \times \mathbf{H}_r(\mathbf{r},t)$

(e) The full calculation reads

$$\begin{aligned} \langle \mathbf{S}_r(\mathbf{r},t)\rangle &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\mathbf{S}_r(\mathbf{r},t)\,\mathrm{d}t && (1) \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\mathbf{E}_r(\mathbf{r},t)\times\mathbf{H}_r(\mathbf{r},t)\,\mathrm{d}t && (2) \\ &= \frac{E_0^2}{\omega_0\mu_0}e^{-2\alpha z}\mathbf{e}_z\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\cos(\beta z-\omega_0 t+\phi)\left[\beta\cos(\beta z-\omega_0 t+\phi)-\alpha\sin(\beta z-\omega_0 t+\phi)\right]\mathrm{d}t \\ &= \frac{E_0^2}{\omega_0\mu_0}e^{-2\alpha z}\mathbf{e}_z\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T}\left[\beta\frac{\cos(2(\beta z-\omega_0 t+\phi))+1}{2} - \alpha\frac{\sin(2(\beta z-\omega_0 t+\phi))}{2}\right]\mathrm{d}t \\ &= \frac{E_0^2}{\omega_0\mu_0}\frac{\beta}{2}e^{-2\alpha z}\mathbf{e}_z && (3) \end{aligned}$$