
Theoretical Optics

Exercise 3

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Drop point: Your tutorial group in ILIAS

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Problem 1. (15 points) Electrostatic stress tensor

Consider two spherical shells with a radius of R are charged with a surface charge density of σ, and are separated by a center-to-center distance $d > 2R$ in free space (see Figure below).

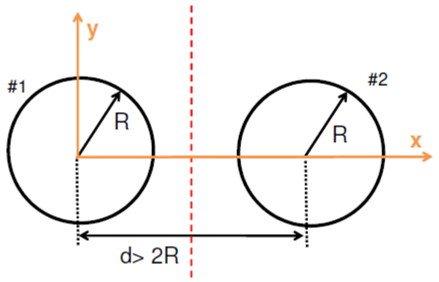

Figure 1: Two identical spherical shells in free space with a surface charge density of σ are depicted in x-y plane.

(a) Show that the total electric field caused by two spherical shells at the planar surface sketched with a red dashed-line ($x = \frac{d}{2}$) in Figure 1 is written as

$$\mathbf{E} = \frac{2\sigma R^2}{\epsilon_0}\left(\frac{1}{d^2/4 + y^2 + z^2}\right)^{3/2}(y\hat{y} + z\hat{z}).$$

(5 points)

(b) Calculate Maxwell's stress tensor $\overleftrightarrow{\mathbf{T}}$ at $x = d/2$. (2 points)

(c) By using the stress tensor $\overleftrightarrow{\mathbf{T}}$ obtained above, show that the force that the shell #1 exerts on the shell #2 is written as

$$\mathbf{F} = \frac{4\pi\sigma^2 R^4}{\epsilon_0 d^2}\hat{\mathbf{x}}.$$

(*Hint*: A proper surface for the surface integration is y-z plane at $x = d/2$.) (6 points)

(d) Verify that the calculated force above is identical to the force between two charged *point* particles when the charge of each particle is $Q = 4\pi R^2\sigma$. (2 points)

Solution to problem 1

(a) The electric force in this problem is determined from Gauss's law. The surface for integration is exactly between the center of the two spheres. This surface close at infinity, where the electric field is zero. Setting the origin of the system at the center of sphere 1, the electric field due to sphere 1 is given by

$$\mathbf{E}_1 = \frac{q}{4\pi\epsilon_0 r^2}\hat{r} = \frac{\sigma R^2}{\epsilon_0 r^2}\hat{r}$$

where $q = \sigma 4\pi R^2$ is the total charge on each sphere .

Writing the electric field in Cartesian coordinate system, and setting the $x = \frac{d}{2}$, the electric field of the the sphere 1 will be

$$\mathbf{E}_1 = \frac{\sigma R^2}{\epsilon_0}(\frac{1}{(d/2)^2 + y^2 + z^2})(\frac{d/2\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{(d/2)^2 + y^2 + z^2}})$$

using $r = \sqrt{(d/2)^2 + y^2 + z^2}$as the distance from the center of sphere 1 to the plane. The field of second sphere is

$$\begin{aligned}\mathbf{E}_2 &= \frac{\sigma R^2}{\epsilon_0}(\frac{1}{(d/2-d)^2 + y^2 + z^2})(\frac{(d/2-d)\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{(d/2-d)^2 + y^2 + z^2}}) \\ &= \frac{\sigma R^2}{\epsilon_0}(\frac{1}{d^2/4 + y^2 + z^2})(\frac{-d/2\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{d^2/4 + y^2 + z^2}})\end{aligned}$$

We have to calculate total field in the system because the stress tensor depends on that. The total field at red-dashed line will be superposition of two fields of two spheres :

$$\mathbf{E}_t = \mathbf{E}_1 + \mathbf{E}_2 = \frac{2\sigma R^2}{\epsilon_0}(\frac{1}{d^2/4 + y^2 + z^2})^{3/2}(y\hat{y} + z\hat{z})$$

This will provide all information for stress tensor

$$\begin{aligned}E_x &= 0 \\ E_y &= \frac{2\sigma R^2}{\epsilon_0}\frac{y}{(d^2/4 + y^2 + z^2)^{3/2}} \\ E_z &= \frac{2\sigma R^2}{\epsilon_0}\frac{z}{(d^2/4 + y^2 + z^2)^{3/2}} \\ E^2 &= \frac{4\sigma^2 R^4}{\epsilon_0^2}\frac{y^2 + z^2}{(d^2/4 + y^2 + z^2)^3}\end{aligned}$$

(b) General form of the stress tensor in free space is

$$T_{ij} \equiv \epsilon_0(E_iE_j - \frac{1}{2}\delta_{ij}E^2) + \frac{1}{\mu_0}(B_iB_j - \frac{1}{2}\delta_{ij}B^2)$$

Here, there are NO magnetic fields in this problem so the electrostatic Maxwell stress tensor is

$$\overleftrightarrow{\mathbf{T}} = \epsilon_0 \begin{bmatrix} E_x^2 - \frac{E^2}{2} & E_xE_y & E_xE_z \\ E_yE_x & E_y^2 - \frac{E^2}{2} & E_yE_z \\ E_zE_x & E_zE_y & E_z^2 - \frac{E^2}{2} \end{bmatrix} = \epsilon_0 \begin{bmatrix} -\frac{E^2}{2} & 0 & 0 \\ 0 & E_y^2 - \frac{E^2}{2} & E_yE_z \\ 0 & E_zE_y & E_z^2 - \frac{E^2}{2} \end{bmatrix},$$

(c) The force in sphere 2 due to the 1 may be written as

$$\mathbf{F} = \oint_s \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{A} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overleftrightarrow{\mathbf{T}} \cdot (-\hat{\mathbf{x}})\mathrm{d}y\mathrm{d}z$$

$$\begin{aligned}\overleftrightarrow{\mathbf{T}}.(-\hat{x}) &= \epsilon_0 \begin{bmatrix} -\frac{E^2}{2} & 0 & 0 \\ 0 & E_y^2 - \frac{E^2}{2} & E_yE_z \\ 0 & E_zE_y & E_z^2 - \frac{E^2}{2} \end{bmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \\ &= \epsilon_0\frac{E^2}{2}\hat{\mathbf{x}} \\ &= \frac{2\sigma^2 R^4}{\epsilon_0}\frac{y^2 + z^2}{(d^2/4 + y^2 + z^2)^3}\hat{\mathbf{x}}\end{aligned}$$

$$\begin{aligned}\mathbf{F} &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overleftrightarrow{\mathbf{T}} \cdot (-\hat{\mathbf{x}})\mathrm{d}y\mathrm{d}z \\ &= \frac{2\sigma^2 R^4}{\epsilon_0}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{y^2 + z^2}{(d^2/4 + y^2 + z^2)^3}\mathrm{d}y\mathrm{d}z\hat{\mathbf{x}}\end{aligned}$$

To solve the integral, consider yz -plane to be in the Cylindrical coordinates system i.e. $\rho = \sqrt{y^2+z^2}$

$$\begin{aligned}
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{y^2+z^2}{(d^2/4+y^2+z^2)^3}\mathrm{d}y\mathrm{d}z &= \int_{0}^{+\infty}\int_{0}^{2\pi} \frac{\rho^2}{(d^2/4+\rho^2)^3}\rho\mathrm{d}\rho\mathrm{d}\phi \\
&= 2\pi\int_{0}^{+\infty} \frac{\rho^2}{(d^2/4+\rho^2)^3}\rho\mathrm{d}\rho \\
&= \pi\int_{0}^{+\infty} \frac{u}{(d^2/4+u)^3}\mathrm{d}u \\
&= \frac{2\pi}{d^2}
\end{aligned}$$

The final solution will be

$$\mathbf{F} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \overleftrightarrow{\mathbf{T}}\cdot(-\hat{\mathbf{x}})\mathrm{d}y\mathrm{d}z = \frac{4\pi\sigma^2R^4}{\epsilon_0 d^2}\hat{\mathbf{x}}$$

(d) The force of two charged particles (with charge $q = 4\pi\sigma R^2$) separated by distance d is

$$\mathbf{F} = \frac{q^2}{4\pi\epsilon_0 d^2}\hat{\mathbf{x}} = \frac{(4\pi\sigma R^2)^2}{4\pi\epsilon_0 d^2}\hat{\mathbf{x}} = \frac{4\pi\sigma^2R^4}{\epsilon_0 d^2}\hat{\mathbf{x}}$$

Problem 2. (9 points) Scalar diffraction theory

Consider the circularly symmetric object shown in Fig. 2. It is infinite in extent in the x-y plane. Its amplitude transmission function is given by

$$t_{\mathrm{A}}(r) = 2\pi J_0(ar) + 4\pi J_0(2ar),$$

where $J_0(x)$ is the zeroth-order Bessel function of the first kind, a is some positive real number signifying a spatial frequency (*i.e.* it has the units of m^{-1}), and $r = \sqrt{x^2+y^2}$ is the radial coordinate in the two-dimensional plane. In scalar approximation, this object is illuminated by a normally incident, unit-amplitude plane wave propagating along z direction, and the paraxial condition is assumed to hold.

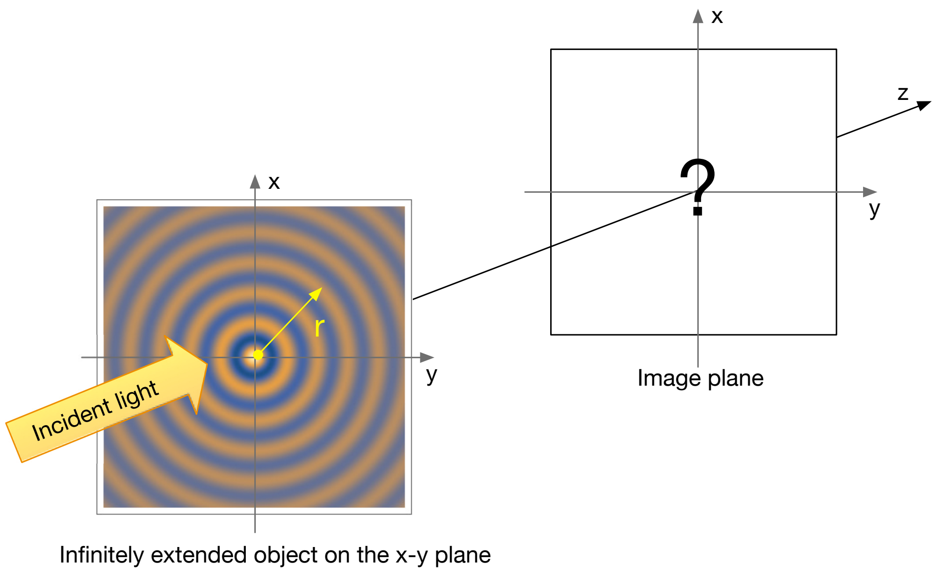

Figure 2: A circularly symmetric object in $x-y$ plane.

(a) Using Fresnel approximation, find the expression of the field distribution in the image plane at z.

(4 points)

(*Hint*: Use the formula for the Fourier transformation ($\mathcal{FT}$) of the Bessel function: $\frac{1}{(2\pi)^2}\iint 2\pi\gamma J_0(\gamma\sqrt{x^2+y^2})e^{-i(\alpha x+\beta y)}\mathrm{d}x\mathrm{d}y = \delta(\sqrt{\alpha^2+\beta^2}-\gamma)$, where γ is a some positive real constant.

Also use the property of the Dirac delta function:
$\mathcal{FT}[\delta(x-x_0)f(x)] = \mathcal{FT}[\delta(x-x_0)f(x_0)]$ where $f(x)$ has no singularity in the whole space.)

(b) Discuss the change of the field's amplitude in the direction of propagation z. (2 points)

(c) At what distances z behind the object, will we find a field distribution that is of the same form as that of the object, up to possible complex constants (*i.e.* ignore the overall phase factor)? (3 points)

Solution to problem 2

(a)

$$\begin{aligned}
u(\mathbf{r}) &= \iint_{-\infty}^{\infty} U(\alpha,\beta;z=0)e^{ikz}e^{-i\frac{z}{2k}(\alpha^2+\beta^2)}e^{i(\alpha x+\beta y)}\mathrm{d}\alpha\mathrm{d}\beta \\
&= \iint_{-\infty}^{\infty}\left(\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} t_{\mathrm{A}}e^{-i(\alpha x'+\beta y')}\mathrm{d}x'\mathrm{d}y'\right)e^{ikz}e^{-i\frac{z}{2k}(\alpha^2+\beta^2)}e^{i(\alpha x+\beta y)}\mathrm{d}\alpha\mathrm{d}\beta \\
&\overset{\mathcal{FT}}{=} \iint_{-\infty}^{\infty}\left(\delta(\sqrt{\alpha^2+\beta^2}-a)+\delta(\sqrt{\alpha^2+\beta^2}-2a)\right)\frac{1}{a}e^{ikz}e^{-i\frac{z}{2k}(\alpha^2+\beta^2)}e^{i(\alpha x+\beta y)}\mathrm{d}\alpha\mathrm{d}\beta \\
&= \iint_{-\infty}^{\infty}\left(\delta(\sqrt{\alpha^2+\beta^2}-a)e^{ikz}e^{-i\frac{z}{2k}a^2}+\delta(\sqrt{\alpha^2+\beta^2}-2a)e^{ikz}e^{-i\frac{z}{2k}4a^2}\right)\frac{1}{a}e^{i(\alpha x+\beta y)}\mathrm{d}\alpha\mathrm{d}\beta \\
&\overset{\mathcal{FT}^{-1}}{=} 2\pi J_0(ar)e^{ikz}e^{-i\frac{z}{2k}a^2}+4\pi J_0(2ar)e^{ikz}e^{-i\frac{z}{2k}4a^2}
\end{aligned}$$

(b) Field distribution depends on z as

$$u(\mathbf{r}) \sim e^{ikz}e^{-i\frac{a^2}{2k}z}.$$

Writing $k = k_r + ik_i$ we can write the above dependence in terms of the amplitude and the oscillating part

$$u(\mathbf{r}) \sim e^{-k_i\left(1+\frac{a^2}{2|k|^2}\right)z}e^{ik_r\left(1-\frac{a^2}{2|k|^2}\right)z}.$$

For positive values of imaginary part of k the amplitude decreases. For negative values of k_i, the amplitude would increase.

(c) For this part we consider real values of k. The condition for z can be written as

$$\begin{aligned}
u(\mathbf{r}) &= e^{ikz}e^{-i\frac{z}{2k}a^2}\left(2\pi J_0(ar)+4\pi J_0(2ar)e^{-i\frac{z}{2k}3a^2}\right) \\
&= e^{ikz}e^{-i\frac{z}{2k}a^2}\left(2\pi J_0(ar)+4\pi J_0(2ar)\right),\ \text{when } e^{-i\frac{z}{2k}3a^2}=1
\end{aligned}$$

That is,

$$\frac{z}{2k}3a^2 = 2\pi m,\ \text{where } m \ \text{ is an integer},$$

so

$$z = \frac{4k}{3a^2}m\pi.$$