Theoretical Optics

Exercise 4

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Drop point: Your tutorial group in ILIAS

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Problem 1. (7 points) Diffraction theory

In paraxial (Fresnel) approximation, any periodic field distribution $u_0(x+d) = u_0(x)$ is reproduced (except for a global phase factor) after propagation lengths z_m that are integer multiples of a distinct length L_T, called *Talbot length*. To prove that, we need to develop the following steps:

(a) Write the exact expression of the transfer function for propagation in homogeneous space. Furthermore, derive the Fresnel transfer function in homogeneous space. (1 point)
(*Hint:* Use the Taylor expansion $\sqrt{1-x} \approx 1 - \frac{1}{2}x$ for $x \ll 1$.)

(b) For an arbitrary field $u_0(x, z=0)$ expanded into a Fourier series, i.e., $u_0(x, z=0) = \sum_n a_n e^{in\frac{2\pi}{d}x}$, calculate the spatial Fourier spectrum $U_0(\alpha, z=0)$.
(*Useful formula:* $\frac{1}{2\pi}\int e^{i\beta x}e^{-i\alpha x}\mathrm{d}x = \delta(\beta-\alpha)$.) (2 points)

(c) Calculate the field $u(x,z)$ by means of the Fresnel transfer function in homogeneous space. (2 points)

(d) Show that the propagation lengths where the field reproduces (up to a global phase factor) are given by

$$z_m = mL_\mathrm{T},$$

where $L_\mathrm{T} = \frac{2d^2}{\lambda_0}$, and $m \in \mathbb{N}$. (2 points)

Solution to problem 1

(a) Without approximation the transfer function reads $H(\alpha,\beta,z) = e^{i\sqrt{k^2-\alpha^2-\beta^2}z}$. If $\alpha^2+\beta^2 \ll k^2$, then

$$H(\alpha,\beta,z) = e^{i\sqrt{k^2-\alpha^2-\beta^2}z} = e^{ikz\sqrt{1-\frac{\alpha^2+\beta^2}{k^2}}} \approx e^{ikz}e^{-i\frac{\alpha^2+\beta^2}{2k}z},$$

that is the Fresnel transfer function.

(b) Calculate the spatial frequency spectrum $U_0(\alpha)$ of $u_0(x) = \sum_n a_n e^{in\frac{2\pi}{d}x} = \sum_n a_n e^{inK_x x}$

$$\begin{aligned} U_0(\alpha, z=0) &= \frac{1}{2\pi}\int u_0(x)e^{-i\alpha x}dx \\ &= \frac{1}{2\pi}\int \sum_n a_n e^{in\frac{2\pi}{d}x}e^{-i\alpha x}dx \\ &= \sum a_n \delta(nK_x - \alpha) \end{aligned}$$

(c) $H_F(\alpha; z) = e^{ik_0 z - i\frac{\alpha^2}{2k_0}z}$ in the paraxial (Fresnel) approximation:

$$\begin{aligned} u(x,z) &= \int U_0(\alpha,0)H_F(\alpha;z)e^{i\alpha x}d\alpha \\ &= e^{ik_0 z}\sum_n \int a_n \delta(nK_x-\alpha)e^{-i\frac{\alpha^2}{2k_0}z}e^{i\alpha x}d\alpha \\ &= e^{ik_0 z}\sum_n a_n e^{-i\frac{n^2K_x^2}{2k_0}z}e^{inK_x x} \end{aligned}$$

(d) If $e^{-i\frac{n^2K_x^2}{2k_0}z} = 1$, then $u(x,z) = e^{ik_0z}u_0(x)$

$$\begin{aligned} e^{-i\frac{n^2K_x^2}{2k_0}z} &= 1 \\ \Rightarrow \frac{n^2K_x^2}{2k_0}z &= 2\pi m \\ \Rightarrow z &= \frac{m}{n^2}\frac{2d^2}{\lambda_0} \end{aligned}$$

$$m = m'n^2 \Rightarrow z = m'\frac{2d^2}{\lambda_0} = m'L_T$$

Problem 2. (10 points) Poisson's spot

Consider a plane wave that propagates in the positive z-direction and which impinges at normal incidence on a circular disk of radius a in free space, as shown in Figure 1. The scalar incident field can be written as $u(\mathbf{r}) = Ae^{ikz}$, where A is a real constant in x'-y' plane, *i.e.* a unit-amplitude illumination.

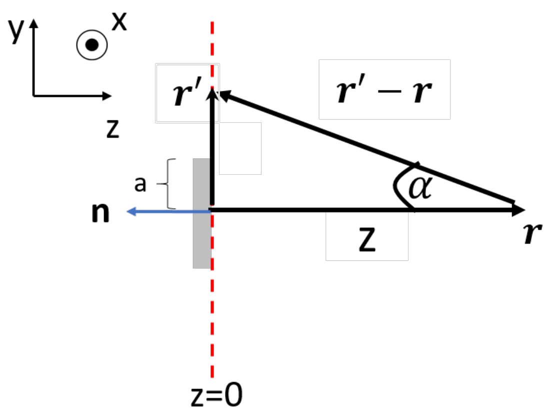

Figure 1: The diffraction problem of Poisson's spot. The main problem consists in the calculation of the field in the positive half-space ($z > 0$) for a circular mask in the x'-y' plane. Please note, the integration in the diffraction integrals goes over the entire x'-y' plane at $z = 0$. Eventually, we are interested here only for the fields at $x = y = 0$, *i.e.*, along the z-axis. The gray box denotes the area of the circular opaque disk that causes Poisson's spot at distance z. The arrow $\mathbf{r}'$ denotes the spatial coordinate of the plane just behind the disk, while the arrow $\mathbf{r}$ represents the point of our observation.

(a) Using the first Rayleigh Sommerfield diffraction formula, show that the field amplitude on the optical axis at the distance z away the center of the disk is given by

$$u_{RS}(0,0,z) = A\frac{z}{r_0}e^{ikr_0},$$

where $r_0^2 = a^2 + z^2$.

(*Hint*: Use the Sommerfeld lemma $\int_a^b f(x)e^{ikx}\mathrm{d}x \approx \frac{f(x)}{ik}e^{ikx}\Big|_a^b$, where the approximation is justified if $f(x)$ is a slowly varying function, or, equivalently, in the limit $k \to \infty$ ($\lambda \to 0$).) (2 points)

(b) By using the Helmholtz and Kirchhoff theorem, prove that for the plane wave incident field, the following Fresnel-Kirchhoff diffraction formula holds

$$u_{FK}(\mathbf{r}) = \frac{1}{2i\lambda}A\iint \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}(1+\cos(\alpha))\,\mathrm{d}^2r',$$

where α is the angle between the outward normal direction of the disk and the vector $\mathbf{r}' - \mathbf{r}$.

(Hint: Use $G(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}$ for Green's function, and the approximation $\frac{1}{|\mathbf{r}'-\mathbf{r}|^2} \approx 0$.) (4 points)

(c) Using the derived Fresnel-Kirchoff diffraction formula, show that the field amplitude at the distance z from the center of the disk can be written as

$$u_{FK}(0,0,z) = \frac{A}{2} e^{ikr_0} \left(1 + \frac{z}{r_0}\right).$$

(*Hint*: Use the approximation $\int_{r_0}^{\infty} e^{ikr} \mathrm{d}r \approx -\frac{e^{ikr_0}}{ik}$ and the Sommerfeld lemma $\int_a^b f(x) e^{ikx} \mathrm{d}x \approx \left.\frac{f(x)}{ik} e^{ikx}\right|_a^b$.)
(4 points)

Solution to problem 2

(a) The Rayleigh Sommerfeld diffraction formula reads as

$$\begin{aligned}
u_{RS}(\mathbf{r}) &= \frac{1}{i\lambda} \iint u(\mathbf{r}') \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} \cos\alpha \, \mathrm{d}^2 r' \\
&= \frac{1}{i\lambda} A \int_{r'=a}^{r'=\infty} \int_{\theta=0}^{\theta=2\pi} \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} \frac{z}{|\mathbf{r}'-\mathbf{r}|} r' \mathrm{d}\theta dr' \\
&= \frac{2\pi}{i\lambda} A \int_{q'=r_0}^{q'=\infty} \frac{e^{ikq'}}{q'} \frac{z}{q'} q' \mathrm{d}q' \\
&= A \frac{z}{r_0} e^{ikr_0},
\end{aligned}$$

where $\mathbf{r} = (0,0,z)$, $\cos\alpha = \frac{z}{|\mathbf{r}'-\mathbf{r}|}$. Above we applied the coordinate transformation via $q' = |\mathbf{r}' - \mathbf{r}|$, by which $r'\mathrm{d}r' = q'\mathrm{d}q'$, and $r_0^2 = a^2 + z^2$, and the use of the Sommerfeld lemma $\int_{r_0}^{\infty} \frac{1}{x} e^{ikx} dx \approx -\frac{1}{ikr_0} e^{ikr_0}$ is made.

(b) The Helmholtz and Kirchoff theorem states that

$$u(\mathbf{r}) = \frac{1}{4\pi} \iint \left[G(\mathbf{r}, \mathbf{r}') \frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} - u(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial \mathbf{n}} \right] \mathrm{d}^2 r'.$$

For the first part (by writing $|\mathbf{r}'| = r'$),

$$\begin{aligned}
\frac{1}{4\pi} \iint \left(\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} \right) \frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} \mathrm{d}^2 r' &= \frac{1}{4\pi} \iint \left(\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} \right) \frac{\partial A}{\partial \mathbf{n}} \mathrm{d}^2 r' \\
&= -\frac{ik}{4\pi} A \iint \left(\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} \right) \mathrm{d}^2 r'.
\end{aligned}$$

For the second term,

$$\begin{aligned}
\frac{1}{4\pi} \iint [-u(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial \mathbf{n}}] &= -\frac{1}{4\pi} A \iint \frac{\partial [\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}]}{\partial \mathbf{n}} \mathrm{d}^2 r' \\
&= -\frac{1}{4\pi} A \iint \cos\alpha \left(\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} \right) \left(ik - \frac{1}{|\mathbf{r}'-\mathbf{r}|} \right) d^2 r' \\
&\approx -\frac{ik}{4\pi} A \iint \cos\alpha \left(\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} \right) \mathrm{d}^2 r'.
\end{aligned}$$

Thus,

$$\begin{aligned}
u(\mathbf{r}) &= -\frac{ik}{4\pi} A \iint \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} (1 + \cos(\alpha)) \, \mathrm{d}^2 r' \\
&= \frac{1}{2i\lambda} A \iint \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} (1 + \cos(\alpha)) \, \mathrm{d}^2 r'.
\end{aligned}$$

(c) The Fresnel-Kirchoff diffraction formula in the optical axis reads as

$$\begin{aligned}
u_{FK}(\mathbf{r}) &= \frac{1}{2i\lambda} A \iint \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\left(1+\cos(\alpha)\right)\mathrm{d}^2 r' \\
&= \frac{1}{2i\lambda} A \iint \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\mathrm{d}^2 r' + \frac{1}{2i\lambda} A \iint \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\cos(\alpha)\mathrm{d}^2 r' \\
&= \frac{1}{2i\lambda} A \int_{r'=a}^{r'=\infty}\int_{\theta=0}^{\theta=2\pi} \frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} r'\mathrm{d}\theta \mathrm{d}r' + \frac{A}{2}\frac{z}{r_0}e^{ikr_0} \\
&= \frac{2\pi}{2i\lambda} A \int_{q'=r_0}^{q'=\infty} \frac{e^{ikq'}}{q'} q'\mathrm{d}q' + \frac{A}{2}\frac{z}{r_0}e^{ikr_0} \\
&= \frac{2\pi}{2i\lambda} A \left(\frac{-e^{ikr_0}}{ik}\right) + \frac{A}{2}\frac{z}{r_0}e^{ikr_0} = \frac{A}{2}e^{ikr_0}(1+\frac{z}{r_0})
\end{aligned}$$

where we used $\int_{r_0}^{\infty} e^{ikr}\mathrm{d}r = -\frac{e^{ikr_0}}{ik}$.

Problem 3. (7 points) Fraunhofer approximation

Compute the diffraction pattern in Fraunhofer approximation for:

(a) A pinhole with radius a
$\left(\textit{Hint}: \int_0^{2\pi} e^{-ix\cos\varphi}\mathrm{d}\varphi = 2\pi J_0(x) \text{ and } \int_0^a J_0(k\rho)\rho\mathrm{d}\rho = a^2\frac{J_1(ka)}{ka}\right)$ (4 points)

(b) A sequence of N pinholes placed along the x-axis with distances of $d > 2a$.
$\left(\textit{Hint}: \sum_{n=0}^{\mathrm{N}-1} x^n = \frac{1-x^{\mathrm{N}}}{1-x}\right)$ (3 points)

Solution to problem 3

(a) The Fourier transform in the referential plane $u_0(x,y)$ is

$$U_0(\alpha,\beta) = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{+\infty} u_0(x,y)e^{-i(\alpha x+\beta y)}\mathrm{d}x\mathrm{d}y.$$

We can write in cylindrical Coordinate i.e. $\mathbf{k}\cdot\mathbf{r} = k\rho\cos\varphi$ and $k = \sqrt{k_x^2+k_y^2} = \sqrt{\alpha^2+\beta^2}$ and using $\int_0^{2\pi} e^{-ixcos\varphi}\mathrm{d}\varphi = 2\pi J_0(x)$, $\int_0^a J_0(k\rho)\rho\mathrm{d}\rho = a^2\frac{J_1(ka)}{ka}$ we calculate the integral

$$(2\pi)^2 U_0(\alpha,\beta) = \int\int_{-\infty}^{+\infty} u_0(x,y)e^{-i(\alpha x+\beta y)}\mathrm{d}x\mathrm{d}y = \int_0^a\int_0^{2\pi} e^{-ik\rho cos\varphi}\rho\mathrm{d}\rho\mathrm{d}\varphi = 2\pi\int_0^a J_0(k\rho)\rho\mathrm{d}\rho = 2\pi a^2\frac{J_1(ka)}{ka}.$$

And finally we have

$$U_0(\alpha,\beta) = \frac{a^2}{2\pi}\frac{J_1(a\sqrt{\alpha^2+\beta^2})}{a\sqrt{\alpha^2+\beta^2}}.$$

The diffraction pattern in Fraunhofer approximation for a pinhole with radius a reads as:

$$\begin{aligned}
u(x,y,z) &= -(2\pi)^2 i\frac{k}{z}U_0(k\frac{x}{z},k\frac{y}{z})e^{ikz}e^{ik\frac{x^2+y^2}{2z}} \\
&= -i2\pi a^2\frac{k}{z}\frac{J_1(\frac{ka}{z}\sqrt{x^2+y^2})}{\frac{ka}{z}\sqrt{x^2+y^2}}e^{ikz}e^{ik\frac{x^2+y^2}{2z}} \\
&= -i2\pi a\frac{J_1(\frac{ka}{z}\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}e^{ikz}e^{ik\frac{x^2+y^2}{2z}}.
\end{aligned}$$

(b) The Fourier transform of $u_0(x,y)$ for a sequence of N pinholes placed along the x-axis with distances of $d > 2a$ equals

$$\begin{aligned}
U_N(\alpha,\beta) &= \sum_{n=0}^{N-1} U_0(\alpha,\beta)e^{i\alpha nd} \\
&= U_0(\alpha,\beta)\sum_{n=0}^{N-1} e^{i\alpha nd} \\
&= U_0(\alpha,\beta)\sum_{n=0}^{N-1} (e^{i\alpha d})^n \\
&= U_0(\alpha,\beta)\frac{1-e^{i\alpha Nd}}{1-e^{i\alpha d}} \\
&= U_0(\alpha,\beta)\frac{e^{\frac{i\alpha Nd}{2}}(e^{-\frac{i\alpha Nd}{2}}-e^{\frac{i\alpha Nd}{2}})}{e^{\frac{i\alpha d}{2}}(e^{-\frac{i\alpha d}{2}}-e^{\frac{i\alpha d}{2}})} \\
&= U_0(\alpha,\beta)\frac{\sin(\alpha Nd/2)}{\sin(\alpha d/2)}e^{i\alpha(N-1)d/2}.
\end{aligned}$$

The diffraction pattern in Fraunhofer approximation reads as:

$$\begin{aligned}
u(x,y,z) &= (2\pi)^2 i\frac{k}{z}U_N(\alpha,\beta)e^{ikz}e^{ik\frac{x^2+y^2}{2z}} \\
&= (2\pi)^2 i\frac{k}{z}U_0(\alpha,\beta)\frac{\sin(\alpha Nd/2)}{\sin(\alpha d/2)}e^{i\alpha(N-1)d/2}e^{ikz}e^{ik\frac{x^2+y^2}{2z}} \\
&= 2\pi i\frac{k}{z}\frac{a}{2\pi}\frac{J_1\left(a\sqrt{\left(k\frac{x}{z}\right)^2+\left(k\frac{y}{z}\right)^2}\right)}{\sqrt{\left(k\frac{x}{z}\right)^2+\left(k\frac{y}{z}\right)^2}}\frac{\sin(k\frac{x}{z}Nd/2)}{\sin(k\frac{x}{z}d/2)}e^{ik\frac{x}{z}(N-1)d/2}e^{ikz}e^{ik\frac{x^2+y^2}{2z}}.
\end{aligned}$$