Theoretical Optics

Exercise 5

C. Rockstuhl, M. Paszkiewicz, N. Perdana, M. Vavilin
Institute of Theoretical Solid State Physics
Karlsruhe Institute of Technology

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Drop point: Your tutorial group in ILIAS

Due Date: July 7th 2022, 16:00

Problem 1. (7 points) Holography

Holography involves the recording and reconstruction of optical waves. Consider a monochromatic optical wave, with complex amplitude in some plane at $z = 0$ is $U(x, y)$. We record a wave with a transparency (hologram) with the complex transmittance $t(x, y)$ (see Figure 1a). Optical detectors used to make transparencies (e.g. photographic emulsion) are responsive to the optical intensity $|U(x, y)|^2$ and therefore do not store the information about the phase $\arg\{U(x, y)\}$. A solution to this problem is mixing of the original wave (object wave) U with a known reference wave U_0 and recording their interference pattern in the $z = 0$ plane.

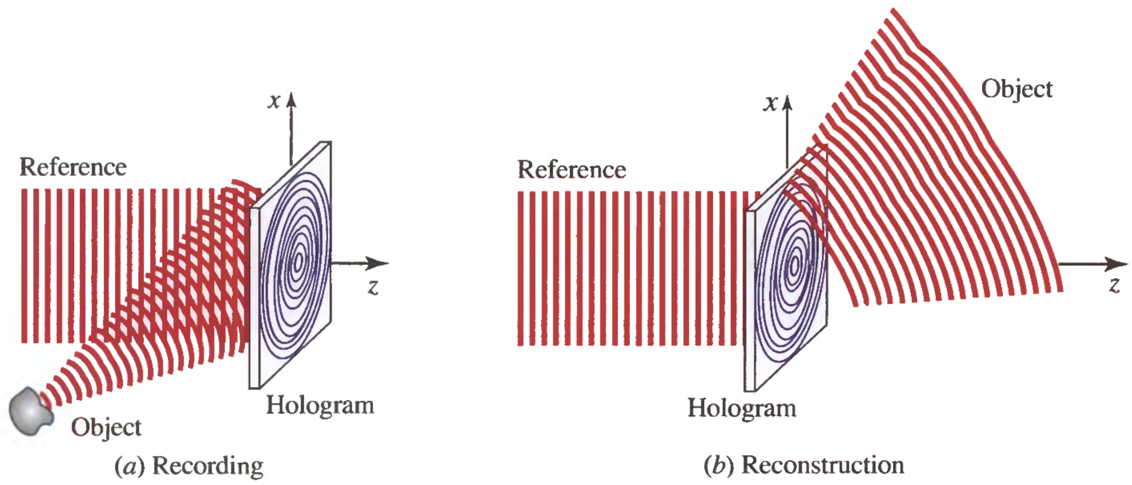

Figure 1: (a) Recording the interference pattern between the object wave and reference wave. (b) Reconstruction of the object wave by illuminating the hologram with a reference wave. Taken from Saleh, B.E. and Teich, M.C., 2007. Fundamentals of photonics. john Wiley & sons.

(a) Show how the information about the amplitude and phase differences between the reference wave and the object wave can be recorded in the transparency. *Hint: The transmittance t is proportional to the intensity* $|U + U_0|$. (2 points)

(b) To reconstruct the object wave, we illuminate the hologram with the reference wave U (Figure 1b). Write the amplitude of resulting wave with a complex amplitude in the hologram plane $z = 0$ in terms of the amplitudes and intensities of the reference and object wave. Explain the meaning of the particular terms in the obtained sum. (2 points)

(c) Assuming that the reference wave is a spherical wave centered about the point $(0, 0, -d)$ and the object wave is a plane wave travelling at an angle θ_x, determine the hologram pattern and the reconstructed wave at $z = 0$. *Hint: Use Fresnel approximation for the spherical wave.* (3 points)

Solution to problem 1

(a) Using the hint we can show that

$$t \sim |U + U_0|^2 = |U|^2 + |U_0|^2 + U_0^* U + U_0 U^* = I + I_0 + U_0^* U + U_0 U^* = I + I_0 + 2\sqrt{I_0 I} \cos(\arg\{U_0\} - \arg\{U\}),$$

where I_0 and I are the intensities of the reference wave and the object wave, respectively, at $z = 0$ plane.

(b) The amplitude U_r of the resulting wave is calculated as:

$$U_r = tU_0 \sim U_0 I_0 + U_0 I + I_0 U + U_0^2 U^*.$$

The first two terms on the right hand side represent the reference wave, modulated by the sum of the intensities of the two waves. The third term is the original wave multiplied by the intensity I_0 of the reference wave. The fourth term is a conjugated version of the original wave modulated by U_0.

(c) The spherical wave can be written as

$$U_0(x, y) = \frac{f_0(x, y)}{|\mathbf{r} - \mathbf{r}_0|} e^{\mathrm{i}k_0|\mathbf{r}-\mathbf{r}_0|} \overset{Fresnel\ approx.}{=} \frac{f_0(x, y)}{|\mathbf{r} - \mathbf{r}_0|} e^{\mathrm{i}k_0 z} e^{\mathrm{i}k_0 \frac{x^2+y^2}{2z}},$$

with $\mathbf{r}_0 = (0, 0, -d)$ and $\mathbf{r} = (x, y, 0)$. The object wave can be expressed as $U(x, y) = f(x, y)e^{\mathrm{i}k_x \sin\theta_x}$. The hologram pattern can be then obtained

$$t \sim |U_0 + U|^2 = \frac{f_0^2}{d^2} + f^2 + \frac{f_0 f}{d^2} e^{-\mathrm{i}k_0 z} e^{\mathrm{i}k_x \sin\theta_x} + \frac{f f_0}{|d|} e^{-\mathrm{i}k_x \sin\theta_x} e^{\mathrm{i}k_0 z} e^{\mathrm{i}k_0 \frac{x^2+y^2}{2d}}.$$

And the reconstructed wave reads

$$U_r(x, y) = tU_0 = \left(\frac{f_0^3}{d^3} + \frac{f^2 f_0}{d}\right) e^{\mathrm{i}k_0 d} e^{\mathrm{i}k_0 \frac{x^2+y^2}{2d}} + \frac{f_0^2 f}{d^3} e^{\mathrm{i}k_x \sin\theta_x} + \frac{f_0^2}{d^2} e^{-\mathrm{i}k_x \sin\theta_x} e^{\mathrm{i}k_0 2\left(d + \frac{x^2+y^2}{2d}\right)}.$$

Problem 2. (17 points) Principal axis of anisotropic crystals

Consider an anisotropic crystal characterized in the laboratory coordinate system by the following relative permittivity tensor

$$\hat{\varepsilon} = \begin{pmatrix} a & 0 & 0 \\ 0 & 1.25a & \alpha a \\ 0 & \alpha a & 1.75a \end{pmatrix},$$

where a is some non-zero number that eventually defines the material properties.
Hint: The dielectric functions are the eigenvalues of the permittivity tensor and the principal axes are defined by the corresponding eigenvectors.

(a) Find all possible solutions for α such that the crystal is uniaxial. (5 points)

(b) For α's obtained from the previous question, rewrite the crystal permittivity tensor in the respective principal axes. (3 points)

(c) For the real positive solution of α from the previous question, write the principal axes of the crystal in terms of the laboratory coordinate basis. (7 points)

(d) How much does the crystal need to be rotated with respect to laboratory coordinate such that we have a diagonalized permittivity tensor in the new coordinate system? (2 points)

Solution to 2

(a) The permittivity of the crystal in its principal axes is equivalent to the eigenvalues of the permittivity tensor. Hence, we have to solve the following algebraic equation.

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1.25a & \alpha a \\ 0 & \alpha a & 1.75a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

To find λ, the determinant of the following matrix should be zero.

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & 1.25a-\lambda & \alpha a \\ 0 & \alpha a & 1.75a-\lambda \end{vmatrix} = 0,$$

which means

$$(a-\lambda)[(1.25a-\lambda)(1.75a-\lambda)-\alpha^2 a^2] = 0.$$

A uniaxial crystal is characterized by two equal entries in the permittivity in principal axes. This implies that λ in the above equation share the same value. There are two possible configuration for this problem:

- Since the first obvious eigenvalue is a, one of the possible scenario is that the solution of $[(1.25a-\lambda)(1.75a-\lambda)-\alpha^2 a^2]$ should be a as well. Plugging this solution for λ into the equation above results in:

$$\begin{aligned} [(1.25a-a)(1.75a-a)-\alpha^2a^2] &= 0 \\ &\downarrow \\ [(0.25a)(0.75a)-\alpha^2a^2] &= 0 \\ &\downarrow \\ [\frac{3}{16}-\alpha^2] &= 0 \\ &\downarrow \\ \alpha &= \pm\sqrt{\frac{3}{16}}. \end{aligned}$$

- Another possible solution requires

$$\begin{aligned} [(1.25a-\lambda)(1.75a-\lambda)-\alpha^2a^2] &= 0 \\ &\downarrow \\ (\frac{5}{4}a-\lambda)(\frac{7}{4}a-\lambda)-\frac{3}{16}a^2 &= 0 \\ &\downarrow \\ \lambda^2-3a\lambda+(\frac{35}{16}-\alpha^2)a^2 &= 0. \end{aligned}$$

  For the above equation to have 2 degenerate solutions, the quadratic equation must satisfy

$$\begin{aligned} 9a^2-4\cdot(\frac{35}{16}-\alpha^2)a^2 &= 0 \\ &\downarrow \\ \frac{36}{16} &= (\frac{35}{16}-\alpha^2) \\ &\downarrow \\ \alpha &= \pm i\sqrt{\frac{1}{16}}. \end{aligned}$$

(b)

- For $\alpha^2 = \frac{3}{16}$, we have eigenvalues $a, a, 2a$, so that

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2a \end{pmatrix}.$$

- For $\alpha^2 = -\frac{1}{16}$, we have eigenvalues $a, 1.5a, 1.5a$, so that

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1.5a & 0 \\ 0 & 0 & 1.5a \end{pmatrix}.$$

Above, the order of eigenvalues in the 3×3 matrix is not yet determined, so you can put them in any order. However, note that the eigenvalues should be ordered in the same order of eigenvectors that span the 3-dimensional coordinate, which we will make clear in the following.

(c) We will use only the case of $\alpha = \sqrt{\frac{3}{16}}$ in what follows. The principal axis are the eigenvectors.

- For the eigenvector $2a$, to find the corresponding eigenvector, we substitute that eigenvalue into the original matrix equation as follows.

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1.25a & \sqrt{\frac{3}{16}}a \\ 0 & \sqrt{\frac{3}{16}}a & 1.75a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 2a \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

from which one can get two equations

$$x_1 = 0, \quad \text{and} \quad 1.25x_2 + \sqrt{\frac{3}{16}}x_3 = 2x_2.$$
$$\downarrow$$
$$x_2 = \frac{1}{\sqrt{3}}x_3.$$

Then the eigenvector (after normalization) has to have a form of

$$\hat{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ \pm\frac{1}{2} \\ \pm\frac{\sqrt{3}}{2} \end{pmatrix}.$$

- For the eigenvector a, to find the corresponding eigenvector, we substitute that eigenvalue into the original matrix equation as follows.

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 1.25a & \sqrt{\frac{3}{16}}a \\ 0 & \sqrt{\frac{3}{16}}a & 1.75a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

from which one can get two equations

$$ax_1 = ax_1 \rightarrow x_1 = \text{undetermined } (\textit{i.e.}, \text{ can be any value})$$
$$1.25x_2 + \sqrt{\frac{3}{16}}x_3 = x_2 \rightarrow x_2 = -\sqrt{3}x_3.$$

The other equation, gives the same result. Then the eigenvector has to have a form of

$$\hat{\mathbf{v}}_{1,2} = \begin{pmatrix} \text{undetermined} \\ -\sqrt{3}x_3 \\ x_3 \end{pmatrix},$$

that constitutes a ring on the surface of a unit sphere in 3-dimensional coordinate, *i.e.*, any point on the surface can be a normalized eigenvector and this is due to the degeneracy. Most conventionally, for simplicity, we choose one eigenvector by putting 1 to the x-component, and 0 to x_3, so that

$$\hat{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and the other is given as

$$\hat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Please note that there there are many possible eigenvectors above. They eventually turn out to span 8 sets of 3-dimensional coordinate systems that are under the π-rotation symmetry to each other. Any of four sets can be solution, however for making a principal axes, we want orthogonality condition and the cross product relation to hold, *i.e.*,

$$\hat{\mathbf{v}}_j \cdot \hat{\mathbf{v}}_k = \delta_{jk}, \text{ and } \hat{\mathbf{v}}_1 \times \hat{\mathbf{v}}_2 = \hat{\mathbf{v}}_3.$$

Therefore the complete basis set is:

$$\text{Set 1} : \hat{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \hat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \ \hat{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix},$$

These sets are also possible:

$$\text{Set 2} : \hat{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \hat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}, \ \hat{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix},$$

$$\text{Set 3} : \hat{\mathbf{v}}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \ \hat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}, \ \hat{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix},$$

$$\text{Set 4} : \hat{\mathbf{v}}_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \ \hat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}, \ \hat{\mathbf{v}}_3 = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix},$$

(d) To diagonalize the permittivity tensor, the rotation should be around the x-axis, we choose the set 1 and 2 here (one is enough for the solution). So, comparing it with the rotation matrix, *i.e.*,

$$\text{Set 1} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \rightarrow \theta = \pi - \frac{\pi}{6},$$

$$\text{Set 2} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \rightarrow \theta = -\frac{\pi}{6}.$$

Any of those can be accepted as the solution.

For your information:

(i) The real part of a permittivity tensor is symmetric.

(ii) The complex permittivity is Hermitian (symmetric real part and antisymmetric imaginary part) when there is no attenuation and dissipation of energy during propagation.

(iii) In case we have attenuation or energy dissipation along propagation, the complex permittivity tensor is symmetric but not Hermitian.

(iv) Complex values for off-diagonal elements of the permittivity tensor does not imply loss as far as it is Hermitian. However, complex values of the diagonal elements imply loss.

(v) If the permittivity tensor is Hermitian: 1) All its eignevalues are real. 2) All its eigenvectors corresponding to non-degenerate eigenvalues are orthogonal and can be made orthonormal. 3) All its eigenvectors corresponding to degenerate eigenvalues can be chosen to be orthogonal or orthonormal.

(vi) If the permittivity is symmetric but not Hermitian, then it might not be diagonalized.