Theoretical Optics

Exercise 6 – Mock Exam

C. Rockstuhl, M. Paszkiewicz, N. Perdana, M. Vavilin
Institute of Theoretical Solid State Physics
Karlsruhe Institute of Technology

___/50 points

Drop point: Your tutorial group in ILIAS

Due Date: July 21st 2022, 16:00

Problem 1. (12 points) Material properties in Maxwell's equations

(a) Write down Maxwell's equations in matter in the presence of both polarization and magnetization but without free charges and their currents. Use a time domain representation. Name all quantities that appear in these equations. (3 points)

(b) From now on, assume non-magnetic, linear, homogenous, isotropic, and dispersive media. The materials are characterized by a response function in time-domain and a susceptibility in frequency domain. How do they link the electric field to the polarization? How are response function and susceptibility connected? Write down mathematical expressions. (3 points)

(c) Define the dielectric function and derive the vectorial Helmholtz equation in frequency domain

$$\Delta \bar{\boldsymbol{E}}(\boldsymbol{r},\omega) + \frac{\omega^2 \epsilon(\omega)}{c_0^2} \bar{\boldsymbol{E}}(\boldsymbol{r},\omega) = 0.$$

Write down mathematical expressions for its non-trivial solutions. (3 points)

(d) From $\bar{n}(\omega) = \sqrt{\epsilon(\omega)}$, find explicit expressions for the real and imaginary parts of the complex refractive index $\bar{n}(\omega) = n(\omega) + iK(\omega)$ as a function of $\Re(\epsilon(\omega))$ and $\Im(\epsilon(\omega))$. (3 points)

Problem 2. (13 points) Far field diffraction from a circular aperture

Consider a circular aperture with a diameter of D, whose transmittance function can be written as

$$t_A(\rho) = \operatorname{circ}\left(\frac{\rho}{D/2}\right),$$

where $\rho = \sqrt{x^2+y^2}$ represents a radius coordinate in the plane of the aperture. Find the Fraunhofer diffraction pattern from the circular aperture known as the Airy pattern. (*Hint*: $\int\int \operatorname{circ}\left(\sqrt{x^2+y^2}\right) e^{-i(\alpha x+\beta y)} dxdy = \frac{2\pi J_1(\sqrt{\alpha^2+\beta^2})}{\sqrt{\alpha^2+\beta^2}}$) (13 points)

Problem 3. (13 points) Propagation in anisotropic media

A plane wave of frequency ω_0 with displacement vector of amplitude $D(z,t) = A\cos(\omega_0 t - kz)$ is incident on a planar uniaxial crystal slab of length L, with $\varepsilon_1(\omega) = \varepsilon_e(\omega)$ and $\varepsilon_2(\omega) = \varepsilon_3(\omega) = \varepsilon_o(\omega)$ (see Fig. 1). The propagation direction $\mathbf{k}$ is orthogonal to the optical axis of the crystal, and the wave is linearly polarized at an angle β with the ordinary polarization direction. The vector $\mathbf{D}(z,t)$ at the input plane can be decomposed into the ordinary ($\hat{\mathbf{o}}$) and extraordinary ($\hat{\mathbf{e}}$) wave directions. It is written as

$$\begin{cases} D_o(z=0,t) = A\cos(\beta)\cos(\omega_0 t) \\ D_e(z=0,t) = A\sin(\beta)\cos(\omega_0 t), \end{cases}$$

each of which will experience a different refractive index while propagating through the slab, and consequently they will travel with different velocities in the medium. This phase lag between the two components will alter the polarization state of the field, *i.e.*, the output field $\mathbf{D}(L,t)$ will be, in general, elliptically polarized.

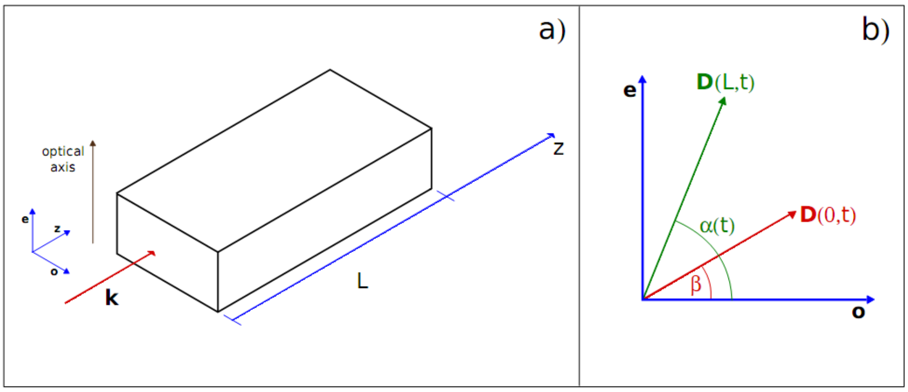

Figure 1: a) Uniaxial crystal slab. b) Direction of the input vector $\mathbf{D}(0,t)$ and output vector $\mathbf{D}(L,t)$ respect to the orthogonal coordinate reference system spanned by the ordinary and extraordinary wave directions.

(a) Write down the expressions of the phases ϕ_o and ϕ_e accumulated by the respective components of the field after propagating through the slab. (3 points)

(b) The vector $\mathbf{D}(L,t)$ makes an angle $\alpha(t)$ with the ordinary polarization direction. Write down the output field $\mathbf{D}(L,t)$ in the basis $\hat{\mathbf{o}}$ and $\hat{\mathbf{e}}$ at the exit face, and prove that

$$\tan(\alpha(t)) = \frac{D_e(L,t)}{D_o(L,t)} = \tan(\beta)\frac{\cos(\omega_0 t - \phi_e)}{\cos(\omega_0 t - \phi_o)}.$$

(3 points)

(c) Find the value of L that gives a phase retardation $\phi_e - \phi_o = 2h\pi$, with $h \in \mathbb{Z}$, and verify that the corresponding value of $\alpha(t)$ is $\alpha(t) = \beta$. What is the state of polarization of the output field $\mathbf{D}(L,t)$ with respect to the incident one?
(*Useful formula:* $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$.) (4 points)

(d) Find the value of L that gives a phase retardation $\phi_e - \phi_o = (2h+1)\pi$, with $h \in \mathbb{Z}$, and show that the corresponding value of $\alpha(t)$ is $\alpha(t) = -\beta$. What is the state of polarization of the output field $\mathbf{D}(L,t)$ with respect to the incident one? (3 points)

Problem 4. (12 points) Temporal coherence of a stochastic process. (It can be solved after the 18 July 2022 lecture.)

Consider the ergodic stochastic process that describes a fluctuation of the output of a laser due to phase noise. The associated electric field can be written as

$$F(t) = \frac{A}{\sqrt{\pi}T} e^{-\frac{t^2}{T^2}} e^{-i(\omega_0 t + \phi)},$$

where T is the pulse duration, and ϕ is a random variable given from a uniform distribution in an interval $[0, 2\pi)$.

(a) Calculate the autocorrelation function $G(\tau) = \langle F^*(t)F(t+\tau)\rangle$. (4 points)

(b) Calculate the power spectral density $S(\omega)$ by means of the Wiener-Khinchin theorem. (4 points)

(c) Calculate the *coherence time* τ_c defined as $\tau_c = \int_{-\infty}^{+\infty} |g(\tau)|^2 \mathrm{d}\tau$, where $g(\tau)$ is the *complex degree of temporal coherence*. (4 points)

Hint: Useful integral identity: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{cx} e^{-\frac{(bx)^2}{2}} \mathrm{d}x = \frac{1}{b} e^{\frac{c^2}{2b^2}}$ for b such that $\Re[b^2] > 0$.

Solution to problem 1

(a)

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\boldsymbol{\nabla}\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\nabla\cdot\boldsymbol{D} = 0\nabla\times\boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t}$$

with

$$\boldsymbol{B} = \mu_0(\boldsymbol{H}+\boldsymbol{M})\boldsymbol{D} = \epsilon_0\boldsymbol{E}+\boldsymbol{P}.$$

Here, $\boldsymbol{E}$ and $\boldsymbol{H}$ represent the vectors of the electric and magnetic fields, while $\boldsymbol{D}$ and $\boldsymbol{B}$ stand for the electric displacement and the magnetic induction, respectively. $\boldsymbol{P}$ is the polarization, and $\boldsymbol{M}$ the magnetization of the medium.

(b) Response function $R(\boldsymbol{r},\tau)$ is given via

$$\boldsymbol{P}(\boldsymbol{r},t) = \epsilon_0\int_{-\infty}^{t} R(\boldsymbol{r},t-t')\boldsymbol{E}(\boldsymbol{r},t')dt'$$

and susceptibility $\chi^{(1)}(\boldsymbol{r},\omega)$ via

$$\bar{\boldsymbol{P}}(\boldsymbol{r},\omega) = \epsilon_0\chi^{(1)}(\boldsymbol{r},\omega)\bar{\boldsymbol{E}}(\boldsymbol{r},\omega).$$

They are conneced by means of a FT:

$$R(t) = \int_{-\infty}^{\infty}\chi^{(1)}(\omega)e^{-i\omega t}d\omega.$$

(c) From the last subsection one gets

$$\bar{\boldsymbol{P}}(\boldsymbol{r},\omega) = \epsilon_0\chi(\omega)\bar{\boldsymbol{E}}(\boldsymbol{r},\omega),$$

hence

$$\bar{\boldsymbol{D}} = \epsilon_0\bar{\boldsymbol{E}}+\bar{\boldsymbol{P}} = \epsilon_0(1+\chi(\omega))\bar{\boldsymbol{E}} = \epsilon_0\epsilon(\omega)\bar{\boldsymbol{P}},$$

where we defined the dielectric function $\epsilon(\omega)$. Using frequency represenation of ME one gets

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\bar{\boldsymbol{E}} - \frac{\omega^2\epsilon(\omega)}{c^2}\bar{\boldsymbol{E}} = 0.$$

Remembering

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\bar{\boldsymbol{E}} = \boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\bar{\boldsymbol{E}} - \Delta\bar{\boldsymbol{E}}$$

one arrives at

$$\Delta\bar{\boldsymbol{E}} + \frac{\omega^2\epsilon(\omega)}{c^2}\bar{\boldsymbol{E}} = 0.$$

The non-trivial solutions are

$$\boldsymbol{E} = \boldsymbol{E}_0 e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}, \quad k = \pm\frac{\omega}{c}\sqrt{\epsilon(\omega)}$$

(d) One should use two equations

$$\bar{n}^2 = \Re(\epsilon) + i\Im(\epsilon), \bar{n} = n + iK.$$

Then, on one hand

$$n^2 - K^2 = \Re(\bar{n}^2) = \Re(\epsilon),$$

and on the other

$$n^2 + K^2 = |\bar{n}|^2 = \Re(\epsilon)^2 + \Im(\epsilon)^2.$$

From the last two equations follow the required

$$n = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\Re(\epsilon)^2+\Im(\epsilon)^2}+\Re(\epsilon)}, K = \frac{1}{\sqrt{2}}\sqrt{\sqrt{\Re(\epsilon)^2+\Im(\epsilon)^2}-\Re(\epsilon)}.$$

Solution to problem 2

(a) In the Fraunhofer approximation, the field can be obtained by

$$u(x, y; z) = \frac{e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)}}{i\lambda z} \int\int u_r(r') e^{-i(k\frac{x}{z}x' + k\frac{y}{z}y')} dx'dy'.$$

Here,

$$u(r') = t_A(\rho)u_0 = u_0 \text{ circ}\left(\frac{\sqrt{x'^2+y'^2}}{D/2}\right) = u_0 \text{ circ}\left(\sqrt{X'^2+Y'^2}\right),$$

where

$$X' = \frac{2x'}{D}, \quad \text{and} \quad Y' = \frac{2y'}{D}.$$

Via the $2D$ Jacobian matrix,

$$dx'dy' = \left(\frac{\partial x'}{\partial X'}\frac{\partial y'}{\partial Y'} - \frac{\partial x'}{\partial Y'}\frac{\partial y'}{\partial Y'}\right) dX'dY' = \frac{D^2}{4} dX'dY'.$$

Then,

$$\begin{aligned}
u(x, y; z) &= \frac{e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)}}{i\lambda z} \frac{D^2}{4} u_0 \int\int \text{circ}\left(\sqrt{X'^2+Y'^2}\right) e^{-i(k\frac{x}{z}\frac{D}{2}X' + k\frac{y}{z}\frac{D}{2}Y')} dX'dY' \\
&= \frac{e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)}}{i\lambda z} \frac{D^2}{4} u_0 \frac{2\pi J_1\left(\sqrt{\alpha^2+\beta^2}\right)}{\sqrt{\alpha^2+\beta^2}}, \text{ where } \alpha = k\frac{x}{z}\frac{D}{2} \text{ and } \beta = k\frac{y}{z}\frac{D}{2} \\
&= \frac{e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)}}{i\lambda z} A u_0 \frac{2J_1\left(\frac{k}{z}\frac{D}{2}\rho\right)}{\frac{k}{z}\frac{D}{2}\rho}, \text{ where } A = \pi D^2/4 \\
&= \frac{e^{ikz} e^{i\frac{k}{2}(x^2+y^2)}}{i} A u_0 \frac{2J_1\left(\frac{k}{z}\frac{D}{2}\rho\right)}{\frac{k}{z}\frac{D}{2}\rho}
\end{aligned}$$

Thus,

$$I(x, y; z) = |u(x, y; z)|^2 = I_0 \left(\frac{2J_1\left(\frac{k}{z}\frac{D}{2}\rho\right)}{\frac{k}{z}\frac{D}{2}\rho}\right)^2, \text{ where } I_0 = (Au_0)^2.$$

Solution to 3

(a) The propagation direction of the plane wave is orthogonal to the optical axis, so the normal modes are excited into the structure. This means that the component D_e will experience a refractive index $n_e(\omega) = \sqrt{\varepsilon_e(\omega)}$, while the component D_o will experience a refractive index $n_o(\omega) = \sqrt{\varepsilon_o(\omega)}$. The phase accumulated by D_e is then

$$\phi_e = k_e L = \frac{2\pi}{\lambda_0} n_e L,$$

while the phase accumulated by D_o is

$$\phi_o = k_o L = \frac{2\pi}{\lambda_0} n_o L.$$

(b) The output field can be written as

$$\begin{cases} D_o(z = L, t) = A\cos(\beta)\cos(\omega_0 t - \phi_o) \\ D_e(z = L, t) = A\sin(\beta)\cos(\omega_0 t - \phi_e). \end{cases}$$

If we perform the ratio of both equations, we obtain

$$\tan(\alpha(t)) = \frac{D_e(L, t)}{D_o(L, t)} = \tan(\beta)\frac{\cos(\omega_0 t - \phi_e)}{\cos(\omega_0 t - \phi_o)} \tag{1}$$

(c) The phase difference $\phi_e - \phi_o$ can be written as

$$\phi_e - \phi_o = \frac{2\pi}{\lambda_0}(n_e - n_o)L. \tag{2}$$

From this expression it follows

$$\phi_e - \phi_o = 2h\pi \quad \Rightarrow \quad L = h\frac{\lambda_0}{(n_e - n_o)}, \quad h \neq 0.$$

In order to calculate $\alpha(t)$, we can write $\phi_e = \phi_o + 2h\pi$, and substitute it in the expression (1), obtaining

$$\tan(\alpha(t)) = \frac{D_e(L,t)}{D_o(L,t)} = \tan(\beta)\frac{\cos(\omega_0 t - \phi_o - 2h\pi)}{\cos(\omega_0 t - \phi_o)}.$$

We can now use the formula $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$, and write

$$\tan(\alpha(t)) = \tan(\beta)\frac{\cos(\omega_0 t - \phi_o)\cos(-2h\pi) - \sin(\omega_0 t - \phi_o)\sin(-2h\pi)}{\cos(\omega_0 t - \phi_o)} = \tan(\beta),$$

which implies $\alpha = \beta$. The output field is linearly polarized and it oscillates in the same polarization plane of the input field.

(d) We can use eq (2), and write

$$\phi_e - \phi_o = (2h+1)\pi \quad \Rightarrow \quad L = (2h+1)\frac{\lambda_0}{2(n_e - n_o)}$$

In order to calculate $\alpha(t)$, we can write $\phi_e = \phi_o + (2h+1)\pi$, and substitute it in the expression (1), obtaining

$$\tan(\alpha(t)) = \frac{D_e(L,t)}{D_o(L,t)} = \tan(\beta)\frac{\cos(\omega_0 t - \phi_o - (2h+1)\pi)}{\cos(\omega_0 t - \phi_o)},$$

that can be rewritten as

$$\tan(\alpha(t)) = \tan(\beta)\frac{\cos(\omega_0 t - \phi_o)\cos((2h+1)\pi) + \sin(\omega_0 t - \phi_o)\sin((2h+1)\pi)}{\cos(\omega_0 t - \phi_o)} = -\tan(\beta),$$

which implies $\alpha(t) = -\beta$. The output field is linearly polarized and the polarization plane is rotated of 2β with respect to the polarization plane of the input field.

Solution to 4

(a)

$$\begin{aligned} G(\tau) &= \langle F^*(t)F(t+\tau)\rangle \\ &= \int_{-\infty}^{+\infty} \left(\frac{A}{\sqrt{\pi}T}\right)^2 e^{-\frac{t^2}{T^2}} e^{-\frac{(t+\tau)^2}{T^2}} e^{i\omega_0 t} e^{-i\omega_0 t} e^{-i\omega_0 \tau} e^{-i\phi} e^{i\phi} \mathrm{d}t \\ &= \frac{A^2}{\pi T^2} e^{-i\omega_0 \tau} e^{-\frac{\tau^2}{T^2}} \int_{-\infty}^{+\infty} e^{-\frac{2t^2}{T^2}} e^{-\frac{2t\tau}{T^2}} \mathrm{d}t = \frac{A^2}{\sqrt{2\pi}T} e^{-\frac{\tau^2}{2T^2}} e^{-i\omega_0 \tau} \end{aligned}$$

(b) From the Wiener-Khinchin theorem we have

$$S(\omega) = \mathcal{F}\left[G(\tau)\right] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(\tau) e^{i\omega\tau} \mathrm{d}\tau$$

If we call $\Gamma(\omega) = \mathcal{F}\left[\frac{A^2}{\sqrt{2\pi}T} e^{-\frac{\tau^2}{2T^2}}\right] = \frac{A^2}{2\pi} e^{-\frac{T^2\omega^2}{2}}$, $\Theta(\omega) = \mathcal{F}\left[e^{i\omega_0\tau}\right] = \delta(\omega - \omega_0)$, we can use the convolution theorem

$$S(\omega) = \Gamma(\omega) * \Theta(\omega) = \frac{A^2}{2\pi} e^{-\frac{T^2(\omega-\omega_0)^2}{2}}.$$

(c) The coherence time is given by

$$\tau_c = \int_{-\infty}^{+\infty} |g(\tau)|^2 \mathrm{d}\tau,$$

where $g(\tau)$ is the complex degree of optical coherence, given by $g(\tau) = e^{-\frac{\tau^2}{2T^2}} e^{-i\omega_0 \tau}$. Thus we have

$$\tau_c = \int_{-\infty}^{+\infty} e^{-\frac{\tau^2}{T^2}} \mathrm{d}\tau = T\sqrt{\pi}.$$