Theoretical Optics

Wave propagation of pulses

Prof. Carsten Rockstuhl

Karlsruhe Institute of Technology

Wave propagation (here for pulses)

in a homogenous isotropic space

$$\mathbf{rot}\,\mathbf{rot}\,\mathbf{E}(\mathbf{r},t) = -\mu_0 \mathbf{rot}\frac{\partial \mathbf{H}(\mathbf{r},t)}{\partial t} = -\mu_0 \frac{\partial}{\partial t}\left[\mathbf{j}(\mathbf{r},t) + \frac{\partial \mathbf{P}(\mathbf{r},t)}{\partial t} + \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}\right]$$

Harmonic ansatz in time and space: $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

$$\frac{\partial}{\partial t} \xrightarrow{\boldsymbol{FT}} -i\omega$$

$$\Delta\bar{\mathbf{E}}(\mathbf{r},\omega) + \frac{\omega^2}{c_0^2}\varepsilon(\omega)\bar{\mathbf{E}}(\mathbf{r},\omega) = 0$$

Spatial Fourier transformation $$\frac{\partial}{\partial \alpha} \xrightarrow{\boldsymbol{FT}} ik_\alpha$$

$$\bar{\mathbf{E}}(\mathbf{k},\omega)\left(-\mathbf{k}^2 + \frac{\omega^2}{c_0^2}\varepsilon(\omega)\right) = 0$$

Dispersion relation: $\mathbf{k}^2 = \frac{\omega^2}{c_0^2}\varepsilon(\omega)$

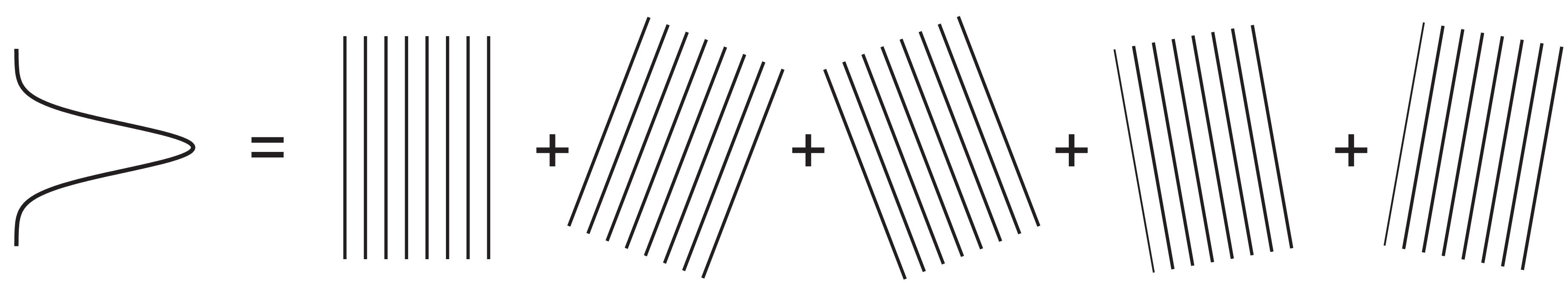
Propagation of a pulsed beam (finite transverse width and finite duration):

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \bar{\mathbf{E}}(\mathbf{k},\omega) e^{i(\mathbf{k}(\omega)\cdot\mathbf{r}-\omega t)} d^3k\, d\omega$$

continuous superposition stationary plane waves
different frequencies and propagation directions

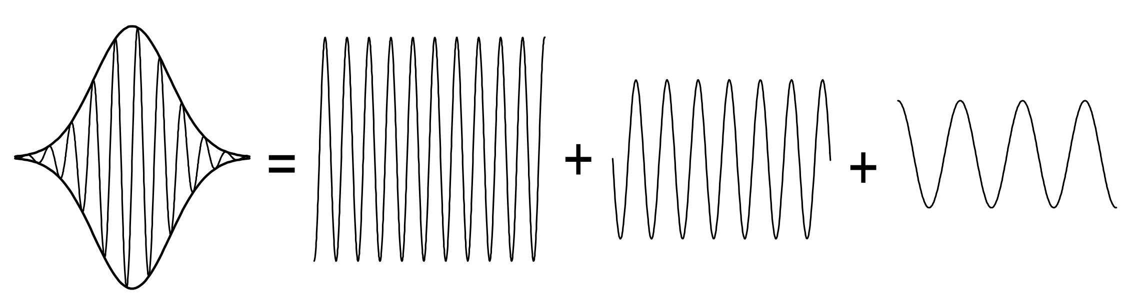
Bundle:
(finite width, time harmonic)

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \bar{\mathbf{E}}(\mathbf{k}) \exp\left[\mathbf{i}\left(\mathbf{kr} - \omega t\right)\right] d^3k$$



Impulse:
(finite length 1 spatial
and 1 temporal dimension)

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \bar{\mathbf{E}}(\omega) \exp\left[\mathbf{i}\left(\mathbf{k}(\omega)\mathbf{r} - \omega t\right)\right] d\omega$$



Simplification: 1D and fixed polarization -> scalar expression

$$u(x,t) = \int_{-\infty}^{\infty} \bar{u}(\omega) e^{i(k(\omega)x - \omega t)} d\omega$$

Pulse envelopes of 10^{-13} s (100fs) $\leq T_0 \leq 10^{-10}$ s (100ps)

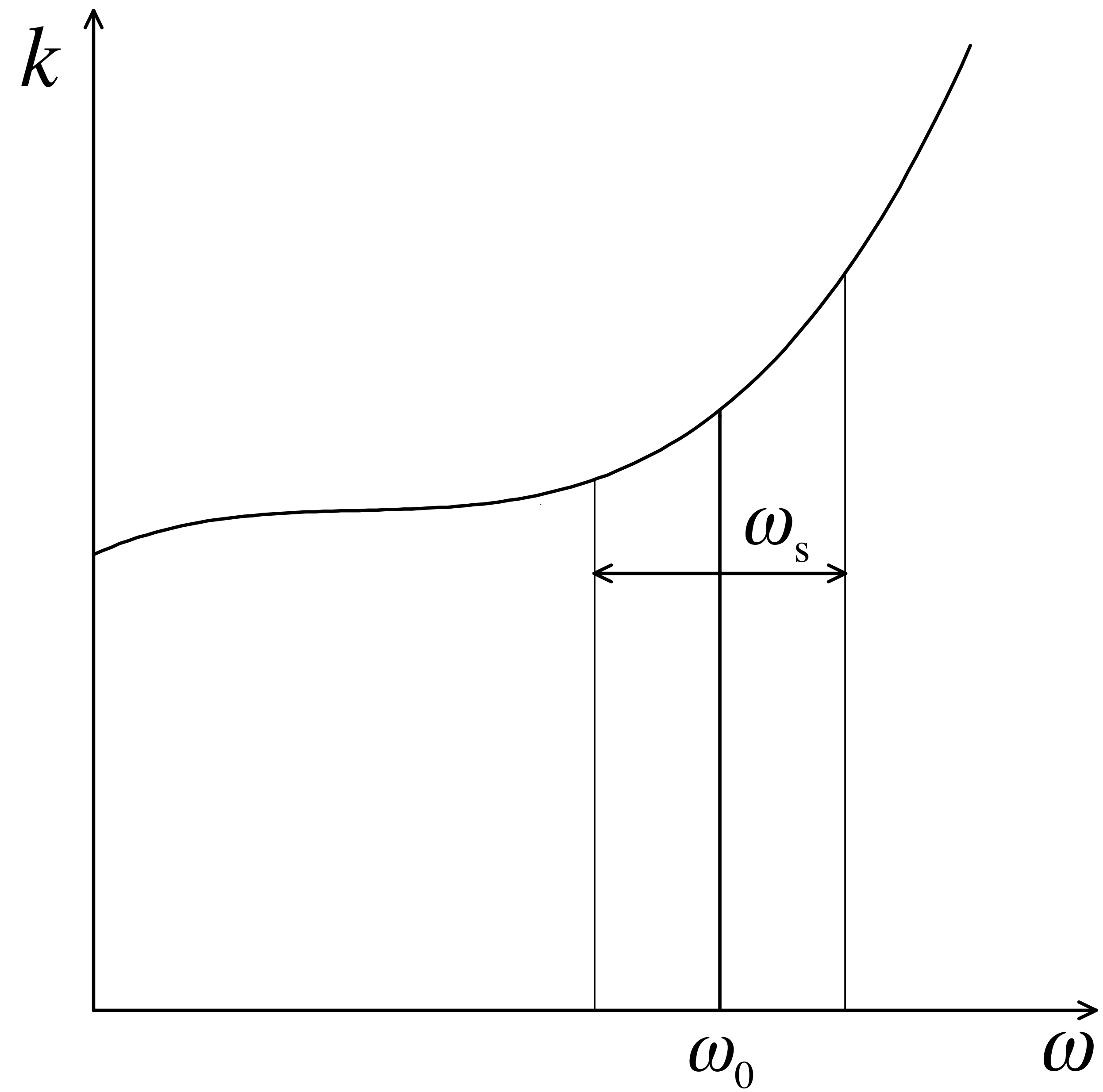
Spectrum of the Gaussian pulse:

$$u(x=0,t) = e^{-i\omega_0 t} e^{-\frac{t^2}{T_0^2}} \longleftrightarrow \bar{u}(x=0,\omega) \propto e^{-\frac{(\omega-\omega_0)^2}{\frac{4}{T_0^2}}}$$

spectral width approximately $\omega_S^2 = 4/T_0^2$

-> 4×10^{10}Hz $\leq \omega_S \leq 4\times10^{13}$Hz

-> small when compared to a center frequency of 4×10^{15} Hz



-> Taylor expansion at $\omega = \omega_0$

$$k(\omega) \approx k(\omega_0) + \left.\frac{\partial k}{\partial \omega}\right|_{\omega_0} (\omega - \omega_0) + \frac{1}{2}\left.\frac{\partial^2 k}{\partial \omega^2}\right|_{\omega_0} (\omega - \omega_0)^2 + \cdots$$

(A) $$k(\omega_0) = k_0 \rightarrow \frac{1}{v_{ph}} = \frac{k_0}{\omega_0} = \frac{n(\omega_0)}{c_0}$$

phase velocity (speed of wave fronts)

(B) $$\frac{1}{v_g} = \left.\frac{\partial k}{\partial \omega}\right|_{\omega_0}$$

group velocity (velocity of the center of the pulse)

$$k(\omega) = \frac{\omega}{c_0} n(\omega) \rightarrow \frac{1}{v_g} = \frac{1}{c_0}\left[n(\omega_0) + \omega_0 \left.\frac{\partial n}{\partial \omega}\right|_{\omega_0}\right]$$

$$v_g = \frac{c_0}{\left[n(\omega_0) + \omega_0 \left.\frac{\partial n}{\partial \omega}\right|_{\omega_0}\right]} = \frac{c_0}{n_g(\omega_0)} = v_{ph} \frac{n(\omega_0)}{n_g(\omega_0)}$$

Group index： $$n_g(\omega_0) = n(\omega_0) + \omega_0 \left.\frac{\partial n}{\partial \omega}\right|_{\omega_0}$$

normal dispersion: $\left.\frac{\partial n}{\partial \omega}\right|_{\omega_0} > 0 \rightarrow n_g > n \rightarrow v_g < v_{ph}$

anomalous dispersion: $\left.\frac{\partial n}{\partial \omega}\right|_{\omega_0} < 0 \rightarrow n_g < n \rightarrow v_g > v_{ph}$

Discussing the implication of that first order term:

$$u(x,t) = \int_{-\infty}^{\infty} \bar{u}(\omega) e^{i\left(\left[k(\omega_0)+\frac{1}{v_g}(\omega-\omega_0)\right]x-\omega t\right)} d\omega$$

$$\longrightarrow \quad \omega - \omega_0 = \bar{\omega}$$

$$v(t) = e^{-\frac{t^2}{T_0^2}} \quad \longleftrightarrow \quad \bar{v}(\bar{\omega}) \propto e^{-\frac{\bar{\omega}^2}{\frac{4}{T_0^2}}}$$

$$u(x,t) = \int_{-\infty}^{\infty} \bar{v}(\bar{\omega}) e^{i\left(\left[k(\omega_0)+\frac{1}{v_g}\bar{\omega}\right]x-(\bar{\omega}+\omega_0)t\right)} d\bar{\omega}$$

$$u(x,t) = e^{i(k(\omega_0)x - \omega_0 t)} \int_{-\infty}^{\infty} \bar{v}(\bar{\omega}) e^{i\left(\frac{1}{v_g}\bar{\omega}x - \bar{\omega}t\right)} d\bar{\omega}$$

$$u(x,t) = e^{i(k(\omega_0)x - \omega_0 t)} \int_{-\infty}^{\infty} \bar{v}(\bar{\omega}) e^{-i\bar{\omega}\left(t - \frac{x}{v_g}\right)} d\bar{\omega}$$

co-moving frame: $\boxed{\tau = t - \frac{x}{v_g}}$

(C) $D_\omega = \left.\frac{\partial^2 k}{\partial \omega^2}\right|_{\omega_0}$

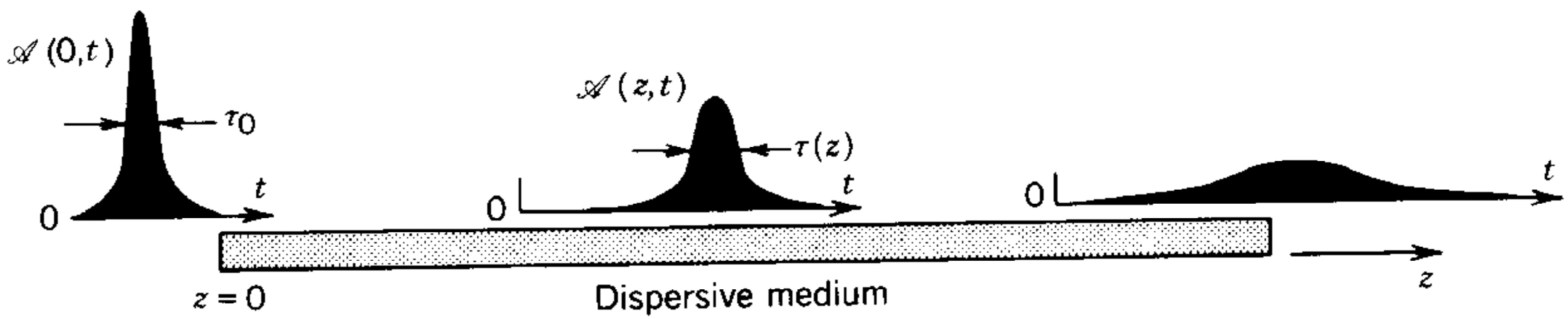
group velocity dispersion (spread of the pulse)

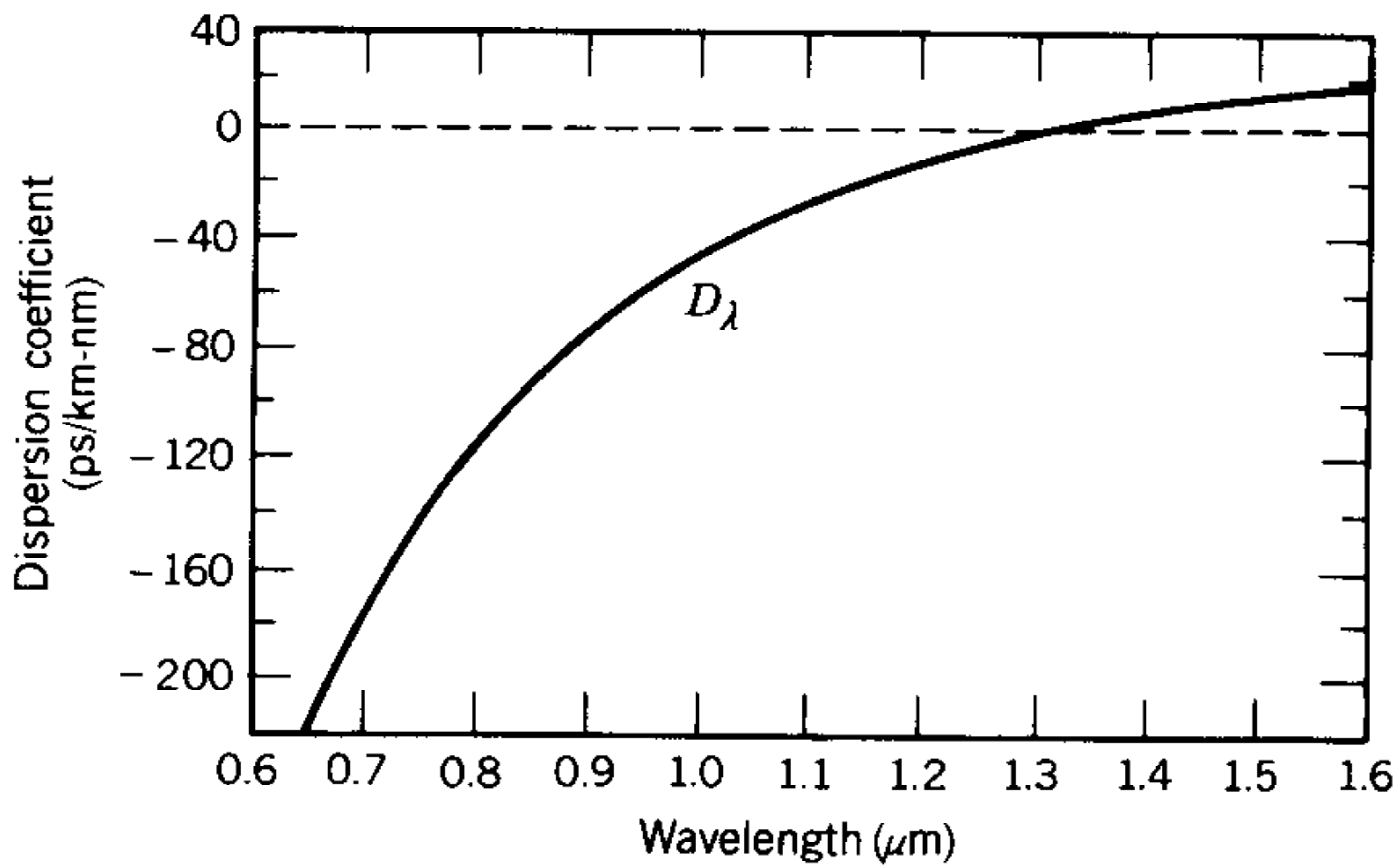
$$D_\omega > 0 \rightarrow \frac{\partial v_g}{\partial \omega} < 0$$

$$D_\omega < 0 \rightarrow \frac{\partial v_g}{\partial \omega} > 0$$

sometimes used in optical communications

$$D_\lambda = \frac{\partial}{\partial\lambda}\left(\frac{1}{v_g}\right) = -\frac{2\pi}{\lambda^2} c D_\omega \longrightarrow$$ change in sign!





Wave propagation of pulses

Prof. Carsten Rockstuhl

Karlsruhe Institute of Technology

Poynting vector and energy balance

Prof. Carsten Rockstuhl

Karlsruhe Institute of Technology

Poynting vector and energy balance

Time averaged Poynting vector

flow of energy expressed using Poynting vector $\mathbf{S}(\mathbf{r},t)$

measurement (detector): $\mathbf{S}(\mathbf{r},t)\cdot\mathbf{n}$

instantaneous energy flux: $\mathbf{S}(\mathbf{r},t) = \mathbf{E}_{\mathrm{r}}(\mathbf{r},t)\times\mathbf{H}_{\mathrm{r}}(\mathbf{r},t)$

note: have to calculate with real valued quantities

time scales:

- fast oscillation of em-field $T_0 = 2\pi/\omega_0 \leq 10^{-14}$ s
- possible pulse duration T_P it holds $T_P \gg T_0$
- duration of measurement T_m it usually holds $T_m \gg T_0$

$T_m \geq T_p$

$$\longrightarrow \quad \mathbf{E}_{\mathrm{r}}(\mathbf{r},t) = \frac{1}{2}\left[\tilde{\mathbf{E}}(\mathbf{r},t)e^{-i\omega_0 t} + c.c.\right]$$

slowly varying envelop

$$
\begin{aligned}
\mathbf{S}(\mathbf{r},t) &= \mathbf{E}_{\mathrm{r}}(\mathbf{r},t)\times\mathbf{H}_{\mathrm{r}}(\mathbf{r},t) \\
&= \tfrac{1}{4}\left[\tilde{\mathbf{E}}(\mathbf{r},t)\times\tilde{\mathbf{H}}^*(\mathbf{r},t) + \tilde{\mathbf{E}}^*(\mathbf{r},t)\times\tilde{\mathbf{H}}(\mathbf{r},t)\right] \\
&\quad + \tfrac{1}{4}\left[\tilde{\mathbf{E}}(\mathbf{r},t)\times\tilde{\mathbf{H}}(\mathbf{r},t)e^{-2i\omega_0 t} + \tilde{\mathbf{E}}^*(\mathbf{r},t)\times\tilde{\mathbf{H}}^*(\mathbf{r},t)e^{2i\omega_0 t}\right] \\
&= \tfrac{1}{2}\Re\left[\tilde{\mathbf{E}}(\mathbf{r},t)\times\tilde{\mathbf{H}}^*(\mathbf{r},t)\right] + \tfrac{1}{2}\Re\left[\tilde{\mathbf{E}}(\mathbf{r},t)\times\tilde{\mathbf{H}}(\mathbf{r},t)\right]\cos 2\omega_0 t \\
&\quad + \tfrac{1}{2}\Im\left[\tilde{\mathbf{E}}^*(\mathbf{r},t)\times\tilde{\mathbf{H}}^*(\mathbf{r},t)\right]\sin 2\omega_0 t
\end{aligned}
$$

detector does not measure fast oscillation but only temporal average

$$\langle\mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2T_m}\int_{t-T_m}^{t+T_m}\mathbf{S}(\mathbf{r},t')dt'$$

fast oscillating terms vanish: $\langle\mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2}\frac{1}{2T_m}\int_{t-T_m}^{t+T_m}\Re\left[\tilde{\mathbf{E}}(\mathbf{r},t')\times\tilde{\mathbf{H}}^*(\mathbf{r},t')\right]dt'$

time harmonic fields: $\tilde{\mathbf{E}}(\mathbf{r},t') = \bar{\mathbf{E}}(\mathbf{r},\omega_0)$ $\quad\tilde{\mathbf{H}}(\mathbf{r},t') = \bar{\mathbf{H}}(\mathbf{r},\omega_0)$

$$I = \langle\mathbf{S}(\mathbf{r},t)\rangle = \frac{1}{2}\Re[\bar{\mathbf{E}}(\mathbf{r},\omega_0)\times\bar{\mathbf{H}}^*(\mathbf{r},\omega_0)]$$

Energy balance / Poynting theorem

expression for dissipation of em-energy density depending on absorption and field magnitude

→ photovoltaics, LEDs

→ starting from Maxwell's curl equations and multiplying fields

$$\mathbf{H}(\mathbf{r},t)\cdot\mathbf{rot}\,\mathbf{E}(\mathbf{r},t)+\mu_0\mathbf{H}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{H}(\mathbf{r},t)=0$$

$$-\varepsilon_0\mathbf{E}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t)+\mathbf{E}(\mathbf{r},t)\cdot\mathbf{rot}\,\mathbf{H}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)\left(\mathbf{j}(\mathbf{r},t)+\frac{\partial}{\partial t}\mathbf{P}(\mathbf{r},t)\right)$$

consider that: $\mathbf{div}\big(\mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)\big)=\mathbf{H}(\mathbf{r},t)\cdot\mathbf{rot}\,\mathbf{E}(\mathbf{r},t)-\mathbf{E}(\mathbf{r},t)\cdot\mathbf{rot}\,\mathbf{H}(\mathbf{r},t)$

$$\mathbf{E}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t)=\frac{1}{2}\frac{\partial}{\partial t}\mathbf{E}^2$$

subtract both equations

$$\frac{1}{2}\varepsilon_0\frac{\partial}{\partial t}\mathbf{E}^2+\frac{1}{2}\mu_0\frac{\partial}{\partial t}\mathbf{H}^2+\mathbf{div}(\mathbf{E}\times\mathbf{H})=-\mathbf{E}\left(\mathbf{j}+\frac{\partial}{\partial t}\mathbf{P}\right)$$

real quantities only as products of fields are taken

$$\frac{1}{2}\varepsilon_0\frac{\partial}{\partial t}\mathbf{E}_{\boldsymbol{r}}^{\mathbf{2}}+\frac{1}{2}\mu_0\frac{\partial}{\partial t}\mathbf{H}_{\boldsymbol{r}}^{\mathbf{2}}+\mathbf{div}(\mathbf{E}_{\boldsymbol{r}}\times\mathbf{H}_{\boldsymbol{r}})=-\mathbf{E}_{\boldsymbol{r}}\left(\mathbf{j}_{\boldsymbol{r}}+\frac{\partial}{\partial t}\mathbf{P}_{\boldsymbol{r}}\right)$$

link between

divergence of Poynting vector

electrical energy density $\frac{1}{2}\varepsilon_0\mathbf{E}_{\boldsymbol{r}}^{\mathbf{2}}$ and magnetic energy density $\frac{1}{2}\mu_0\mathbf{H}_{\boldsymbol{r}}^{\mathbf{2}}$

and the source/sinks of em-energy density to sources

stationary fields:

$$\mathbf{E}_{\mathrm{r}}(\mathbf{r},t)=\frac{1}{2}\left[\bar{\mathbf{E}}(\mathbf{r},\omega_0)e^{-i\omega_0 t}+c.c.\right]$$

$$\mathbf{H}_{\mathrm{r}}(\mathbf{r},t)=\frac{1}{2}\left[\bar{\mathbf{H}}(\mathbf{r},\omega_0)e^{-i\omega_0 t}+c.c.\right]$$

time averaged quantities

lhs.: $\langle \frac{1}{2}\varepsilon_0 \frac{\partial}{\partial t}\mathbf{E}_r^2 + \frac{1}{2}\mu_0 \frac{\partial}{\partial t}\mathbf{H}_r^2 + \mathbf{div}(\mathbf{E}_r \times \mathbf{H}_r)\rangle = \frac{1}{2}\mathbf{div}(\Re[\bar{\mathbf{E}}(\mathbf{r},\omega_0)\times\bar{\mathbf{H}}^*(\mathbf{r},\omega_0)]) = \mathbf{div}\,\langle \mathbf{S}(\mathbf{r},t)\rangle$

time derivative cancels

stationary terms

rhs.: $-\langle\left(\mathbf{j}_r + \frac{\partial}{\partial t}\mathbf{P}_r\right)\mathbf{E}_r\rangle$

$$= -\frac{1}{4}\langle\left[\sigma(\omega_0)\bar{\mathbf{E}}e^{-i\omega_0 t} - i\omega_0\varepsilon_0\chi(\omega_0)\bar{\mathbf{E}}e^{-i\omega_0 t} + c.c.\right]\left[\bar{\mathbf{E}}(\mathbf{r},\omega_0)e^{-i\omega_0 t} + c.c.\right]\rangle$$

$$= -\frac{1}{4}\langle\left[-i\omega_0\varepsilon_0\left(\chi(\omega_0) + i\frac{\sigma(\omega_0)}{\omega_0\varepsilon_0}\right)\bar{\mathbf{E}}e^{-i\omega_0 t} + c.c.\right]\left[\bar{\mathbf{E}}(\mathbf{r},\omega_0)e^{-i\omega_0 t} + c.c.\right]\rangle$$

$$= \frac{1}{4}i\omega_0\varepsilon_0[\varepsilon(\omega_0) - 1]\bar{\mathbf{E}}\bar{\mathbf{E}}^* + c.c. = \frac{1}{4}i\omega_0\varepsilon_0[\varepsilon'(\omega_0) - 1 + i\varepsilon''(\omega_0)]\bar{\mathbf{E}}\bar{\mathbf{E}}^* + c.c.$$

$$= -\frac{1}{2}\omega_0\varepsilon_0\varepsilon''(\omega_0)\bar{\mathbf{E}}\bar{\mathbf{E}}^*$$

$$\longrightarrow \boxed{\mathbf{div}\,\langle \mathbf{S}(\mathbf{r},t)\rangle = -\frac{1}{2}\omega_0\varepsilon_0\varepsilon''(\omega_0)\bar{\mathbf{E}}\bar{\mathbf{E}}^*}$$

non-stationary fields: $\mathbf{E}_\mathrm{r}(\mathbf{r},t) = \frac{1}{2}\left[\tilde{\mathbf{E}}(\mathbf{r},t)e^{-i\omega_0 t} + c.c.\right]$

$$\frac{1}{4}\frac{\partial}{\partial t}\left\{\varepsilon_0\frac{\partial[\omega_0\varepsilon'(\mathbf{r},\omega_0)]}{\partial\omega_0}\left|\tilde{\mathbf{E}}(\mathbf{r},t)\right|^2 + \mu_0\left|\tilde{\mathbf{H}}(\mathbf{r},t)\right|^2\right\} + \mathbf{div}\,\langle \mathbf{S}(\mathbf{r},t)\rangle = -\frac{1}{2}\omega_0\varepsilon_0\varepsilon''(\omega_0)\left|\tilde{\mathbf{E}}(\mathbf{r},t)\right|^2$$

Poynting vector and energy balance

Prof. Carsten Rockstuhl

Karlsruhe Institute of Technology