Theoretical Optics

Pulse propagation and Poynting Theorem

Lecture 03

Prof. Carsten Rockstuhl

Karlsruhe Institute of Technology

Do we consider spatial dispersion?

Nonlocal material response

$$\mathbf{J}(\mathbf{r},\omega) = \int_V \overline{\overline{R}}(\mathbf{r},\mathbf{r}',\omega)\mathbf{E}(\mathbf{r}',\omega)dV'$$

Homogenous material

$$\mathbf{J}(\mathbf{r},\omega) = \int_V \overline{\overline{R}}(\mathbf{r}-\mathbf{r}',\omega)\mathbf{E}(\mathbf{r}',\omega)dV'$$

Spatial Fourier domain

$$\mathbf{J}(\mathbf{k},\omega) = \tilde{R}(\mathbf{k},\omega)\mathbf{E}(\mathbf{k},\omega)$$

Taylor expansion

$$\tilde{R}(\mathbf{k},\omega) \approx \tilde{R}_{ij}(\mathbf{k}=0,\omega) + \left.\frac{\partial \tilde{R}_{ij}(\mathbf{k},\omega)}{\partial k_k}\right|_{\mathbf{k}=0} k_k + \frac{1}{2}\left.\frac{\partial \tilde{R}_{ij}(\mathbf{k},\omega)}{\partial k_k \partial k_l}\right|_{\mathbf{k}=0} k_k k_l$$

$$= -i\omega[a_{ij}(\omega) + a_{ijk}(\omega)k_k + a_{ijkl}(\omega)k_k k_l]$$

Constitutive relation in Fourier space

$$\boxed{J_i(\mathbf{k},\omega) \approx -i\omega \left[a_{ij}(\omega)E_j(\mathbf{k},\omega) + a_{ijk}(\omega)k_k E_j(\mathbf{k},\omega) + a_{ijkl}(\omega)k_j k_k E_j(\mathbf{k},\omega)\right]}$$

Constitutive relation in real space

$$J_i(\mathbf{r},\omega) \approx -i\omega \left[a_{ij}(\omega)E_j(\mathbf{r},\omega) + a_{ijk}(\omega)\partial_k E_j(\mathbf{r},\omega) + a_{ijkl}(\omega)\partial_l\partial_k E_j(\mathbf{r},\omega)\right]$$

$$\boxed{\begin{gathered} D_i(\mathbf{r},\omega) = \left[\varepsilon_0\delta_{ij} + a_{ij}(\omega)\right]E_j(\mathbf{r},\omega) + a_{ijk}(\omega)\partial_k E_j(\mathbf{r},\omega) + a_{ijkl}(\omega)\partial_l\partial_k E_j(\mathbf{r},\omega) \\ \mathbf{H}(\mathbf{r},\omega) = \mu_0^{-1}\mathbf{B}(\mathbf{r},\omega) \end{gathered}}$$

We can shuffle electric and magnetic response!

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} + \mathbf{J}_{\mathrm{ext}}(\mathbf{r},t) \qquad \nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho_{\mathrm{ext}}(\mathbf{r},t)$$

$$\mathbf{D}'(\mathbf{r},t) = \mathbf{D}(\mathbf{r},t) + \nabla \times \mathbf{Q}(\mathbf{r},t), \text{ and } \mathbf{H}'(\mathbf{r},t) = \mathbf{H}(\mathbf{r},t) + \frac{\partial \mathbf{Q}(\mathbf{r},t)}{\partial t}$$

Think of Q as a gauge function; leaves E and B unaffected!

Gauge away to some elements of the first order term

$$Q_i = \frac{1}{4}\varepsilon_{jlm}(a_{ilm} - a_{iml} + a_{mli})E_j$$

Bianisotropic media with weak spatial dispersion

$$\boxed{D_i = \varepsilon_0\varepsilon_{ij}E_j + i\xi_{ij}B_j + a_{ijkl}\partial_l\partial_k E_j, \quad H_i = \mu_0^{-1}B_i + \xi_{ji}E_j}$$

$$\varepsilon_0\varepsilon_{ij} = \varepsilon_0\delta_{ij} + a_{ij}$$

local response

$$\xi_{ij} = \frac{\omega}{4}\varepsilon_{jlm}(a_{ilm} - a_{iml} + a_{mli})$$

bianisotropic response (electro-magnetic coupling)

$a_{ijkl}(\omega)$ weak non locality

- multiple solutions to the wave equation
- additional boundary conditions are needed
- carefully look for traces of such non locality

How multiply curl of Maxwell's equations to come to Poynting theorem?

Maxwell equations

$$\mathbf{rot}\,\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \qquad \mathbf{div}\,\mathbf{D}(\mathbf{r},t) = \rho_{\mathtt{ext}}(\mathbf{r},t)$$

$$\mathbf{rot}\,\mathbf{H}(\mathbf{r},t) = \mathbf{j}_{\mathtt{makr}}(\mathbf{r},t) + \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}, \qquad \mathbf{div}\,\mathbf{B}(\mathbf{r},t) = 0$$

in optics

$$\mathbf{rot}\,\mathbf{E}(\mathbf{r},t) = -\mu_0 \frac{\partial \mathbf{H}(\mathbf{r},t)}{\partial t}, \qquad \varepsilon_0 \mathbf{div}\,\mathbf{E}(\mathbf{r},t) = -\mathbf{div}\,\mathbf{P}(\mathbf{r},t)$$

$$\mathbf{rot}\,\mathbf{H}(\mathbf{r},t) = \mathbf{j}_{\mathtt{cond}}(\mathbf{r},t) + \frac{\partial \mathbf{P}(\mathbf{r},t)}{\partial t} + \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}, \qquad \mathbf{div}\,\mathbf{H}(\mathbf{r},t) = 0$$

straight multiplication

(1) $$\mathbf{H}\cdot\mathbf{rotE} + \mu_0 \mathbf{H}\cdot\frac{\partial}{\partial t}\mathbf{H} = 0$$

(2) $$-\varepsilon_0 \mathbf{E}\cdot\frac{\partial}{\partial t}\mathbf{E} + \mathbf{E}\cdot\mathbf{rotH} = \mathbf{E}\cdot\left(\mathbf{j} + \frac{\partial}{\partial t}\mathbf{P}\right)$$

using

$$\mathbf{div}\left(\mathbf{E}\times\mathbf{H}\right) = \mathbf{H}\cdot\mathbf{rotE} - \mathbf{E}\cdot\mathbf{rotH}$$

$$\mathbf{E}(\mathbf{r},t)\cdot\frac{\partial}{\partial t}\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\frac{\partial}{\partial t}\mathbf{E}^2$$

subtracting 1 - 2

$$\boxed{\frac{1}{2}\varepsilon_0\frac{\partial}{\partial t}\mathbf{E}^2 + \frac{1}{2}\mu_0\frac{\partial}{\partial t}\mathbf{H}^2 + \mathbf{div}(\mathbf{E}\times\mathbf{H}) = -\mathbf{E}\left(\mathbf{j} + \frac{\partial}{\partial t}\mathbf{P}\right)}$$