Theoretical Optics

Maxwell’s stress tensor

Introduction

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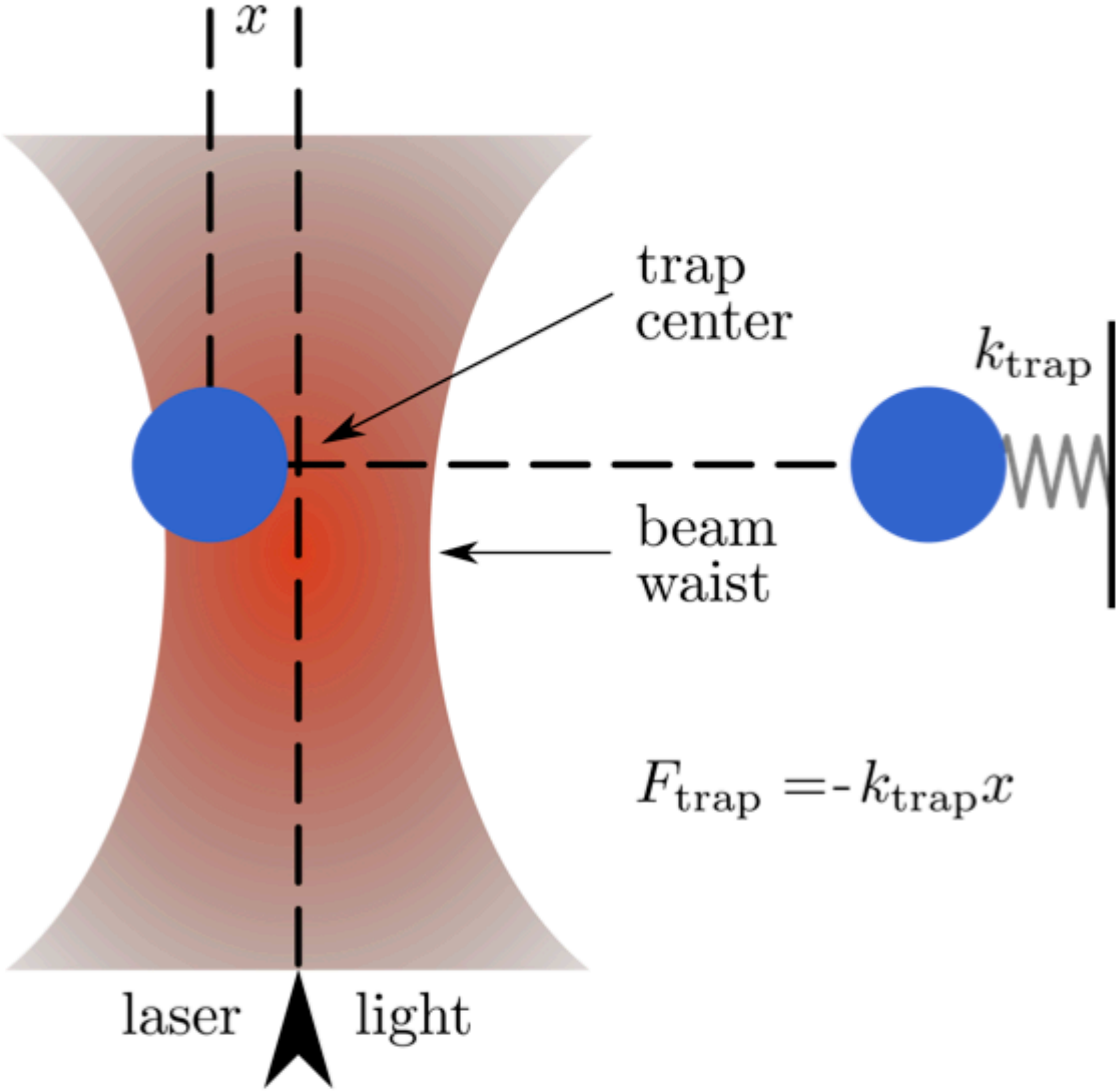
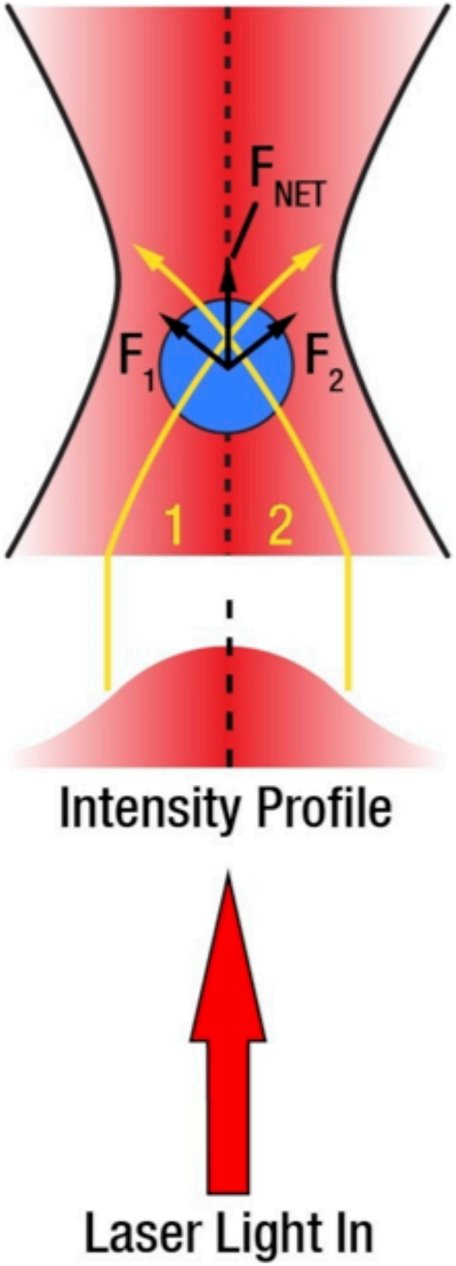
Nobel Prize in Physics 2018

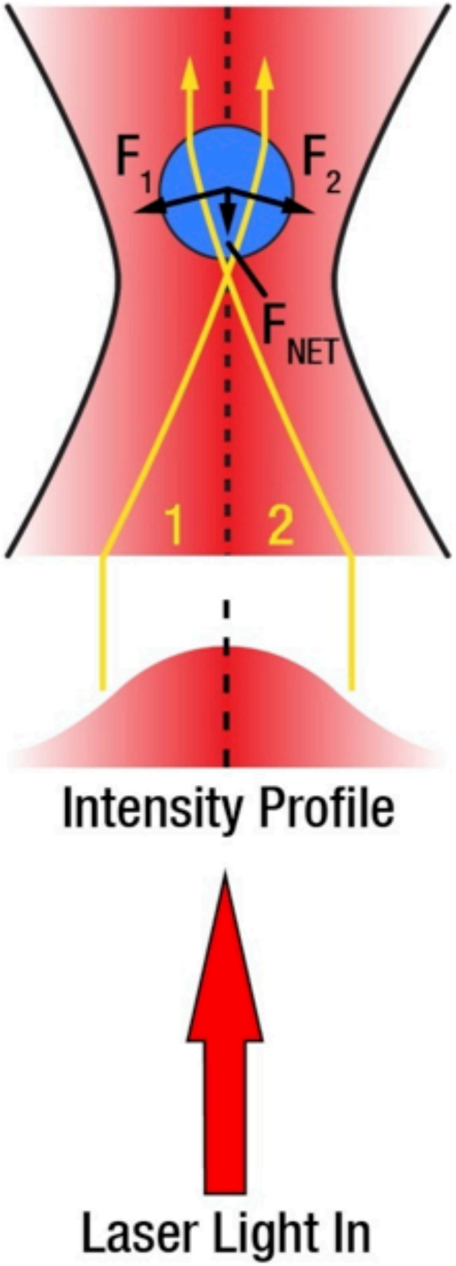


one half to Arthur Ashkin

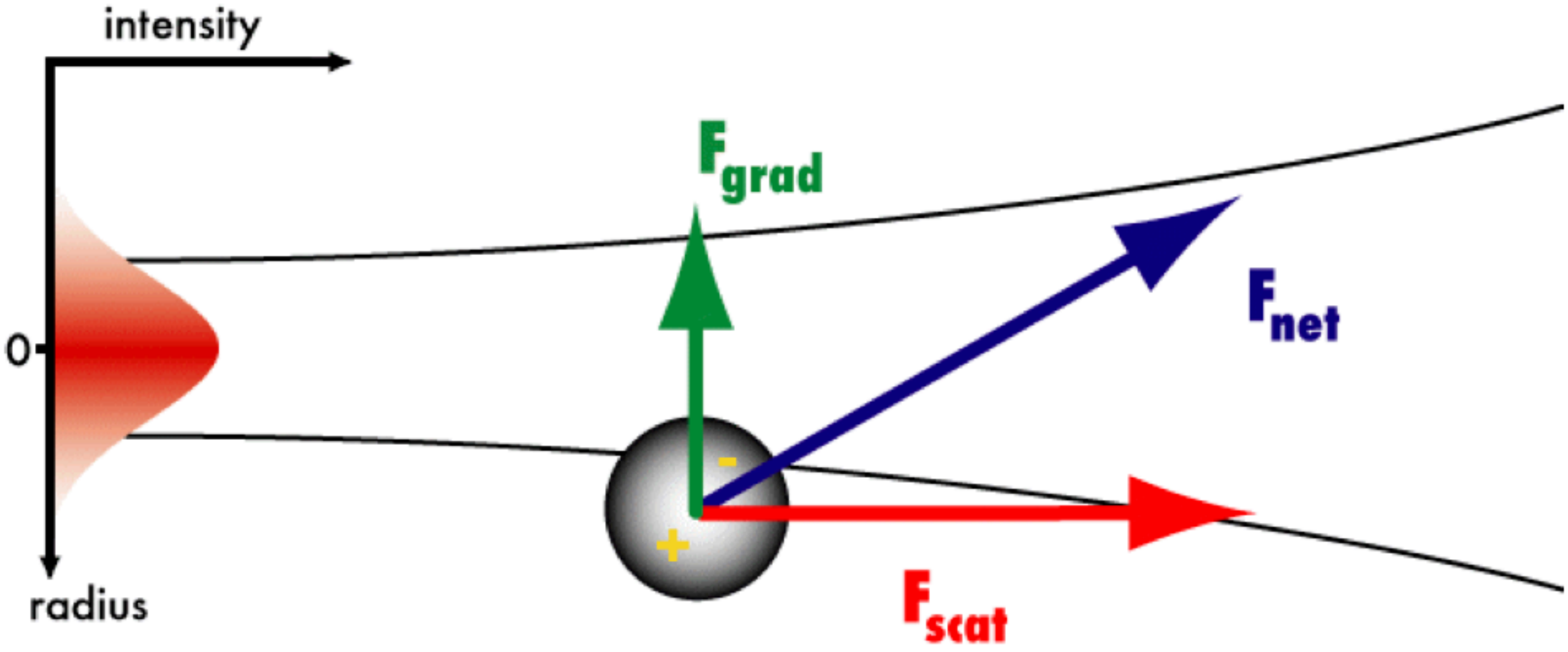
"for the optical tweezers and their application to biological systems"

Ray-optical explanation





Dipole approximation



Force on one charge: (Lorentz force)

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

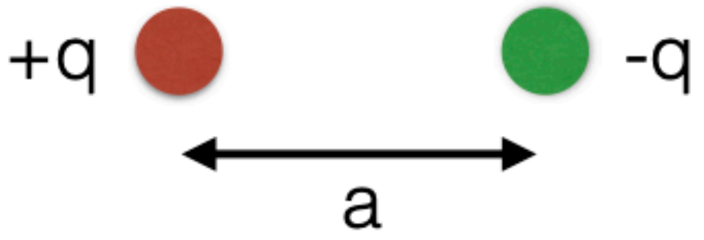
(one charge)

$$= q\mathbf{E}_1 + q\frac{d\mathbf{r}_1}{dt} \times \mathbf{B}$$

Dipole approximation

Dipole moment of two charges:
(opposite sign)

$$\mathbf{p} = q\mathbf{a}$$



Dipole moment of a dipole:

$$\mathbf{p} = \lim_{\substack{\mathbf{a}\to 0 \\ q\to\infty}} q\mathbf{a}$$

Force on two such charges:

$$\mathbf{F} = q\left(\mathbf{E}_1(\mathbf{r}) - \mathbf{E}_2(\mathbf{r}) + \frac{d\left(\mathbf{r}_1 - \mathbf{r}_2\right)}{dt} \times \mathbf{B}\right)$$

$$= q\left(\mathbf{E}_1(\mathbf{r}) + \left(\left(\mathbf{r}_1 - \mathbf{r}_2\right) \cdot \nabla\right)\mathbf{E} - \mathbf{E}_1(\mathbf{r}) + \frac{d\left(\mathbf{r}_1 - \mathbf{r}_2\right)}{dt} \times \mathbf{B}\right)$$

Introducing the dipole moment

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\,\mathbf{E} + \frac{d\mathbf{p}}{dt}\times\mathbf{B}$$

with $\mathbf{p} = \alpha\mathbf{E}$

$$= \alpha\left[(\mathbf{E}\cdot\nabla)\,\mathbf{E} + \frac{d\mathbf{E}}{dt}\times\mathbf{B}\right]$$

using: $(\mathbf{E}\cdot\nabla)\,\mathbf{E} = \nabla\left(\frac{1}{2}E^2\right) - \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E})$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\begin{aligned}\mathbf{F} &= \alpha\left[\frac{1}{2}\nabla E^2 - \mathbf{E}\times(\boldsymbol{\nabla}\times\mathbf{E}) + \frac{d\mathbf{E}}{dt}\times\mathbf{B}\right]\\ &= \alpha\left[\frac{1}{2}\nabla E^2 - \mathbf{E}\times\left(-\frac{d\mathbf{B}}{dt}\right) + \frac{d\mathbf{E}}{dt}\times\mathbf{B}\right]\\ &= \alpha\left[\frac{1}{2}\nabla E^2 + \frac{d}{dt}(\mathbf{E}\times\mathbf{B})\right]\end{aligned}$$

Starting from the expression of the force

$$\vec{F} = \left(\vec{p}\cdot\nabla\right)\vec{E} + \dot{\vec{p}}\times\vec{B}$$

in component notation:

$$\vec{F} = \sum_{i=x,y,z} p_i \nabla E_i + \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{p}\times\vec{B}\right)$$

last term vanishes when time-averaged:

$$\left\langle \vec{F} \right\rangle = \sum_{i=x,y,z} \left\langle p_i(t)\nabla E_i(t)\right\rangle$$

complex notation:

(underlined: complex amplitudes)

$$\vec{E}(\vec{r},t) = \mathrm{Re}\left\{\underline{\vec{E}}(\vec{r})e^{-i\omega t}\right\}$$

$$\vec{B}(\vec{r},t) = \mathrm{Re}\left\{\underline{\vec{B}}(\vec{r})e^{-i\omega t}\right\}$$

linear response:

$$\vec{p}(t) = \mathrm{Re}\left\{\underline{\vec{p}}e^{-i\omega t}\right\}$$

$$\underline{\vec{p}} = \alpha(\omega)\underline{\vec{E}}(\vec{r}_0)$$

force based on complex amplitudes

$$\left\langle \vec{F} \right\rangle = \sum_{i=x,y,z} \frac{1}{2} \mathrm{Re}\left\{ \underline{p}_i^* \nabla \underline{E}_i \right\} = \sum_{i=x,y,z} \frac{1}{2} \mathrm{Re}\left\{ \alpha(\omega) \underline{E}_i\,(\vec{r}) \partial^i \underline{E}_i^*\,(\vec{r}) \right\}$$

for a principle propagation direction:

$$\underline{\vec{E}}(\vec{r}) = \vec{E}_0(\vec{r}) e^{i\vec{k}\cdot\vec{r}}$$

$$\left\langle \vec{F} \right\rangle = \frac{1}{4} \mathrm{Re}\{\alpha(\omega)\} \nabla \left| \vec{E}_0 \right|^2 + \frac{1}{2} \vec{k} \mathrm{Im}\{\alpha(\omega)\} \left| \vec{E}_0 \right|^2 - \frac{1}{2} \mathrm{Im}\{\alpha(\omega)\} \mathrm{Im}\left\{ \vec{E}_0 \cdot \nabla \vec{E}_0^* \right\}$$

gradient force

absorption-plus-scattering longitudinal component of the force (loss/transfer of momentum from the incident light to the particle)

vanishing term when field amplitude or polarisability are real-valued

specification to small particles

$$\alpha(\omega) = \frac{\alpha_0(\omega)}{1-(2/3)ik^3\alpha_0(\omega)}$$

$$\alpha_0(\omega) = a^3(\varepsilon - 1)/(\varepsilon + 2)$$

$$\varepsilon = \varepsilon_{\text{p}}/\varepsilon_{\text{m}}$$

Gradient force: take the real part of polarisability:

$$\left\langle \vec{F}_{\text{grad}} \right\rangle = 4\pi\varepsilon_{\text{m}} a^3 \left(\frac{\varepsilon-1}{\varepsilon+2}\right)\frac{1}{2}\nabla E_0^2 = 4\pi n_{\text{m}}^2 \varepsilon_0 a^3 \left(\frac{m^2-1}{m^2+2}\right)\frac{1}{2}\nabla E_0^2 = 4\pi n_{\text{m}}^2 \varepsilon_0 a^3 \left(\frac{m^2-1}{m^2+2}\right)\frac{1}{2}\nabla I(\vec{r})$$

$$m = n_{\text{p}}/n_{\text{m}} \qquad \langle \vec{E}^2(\vec{r},t)\rangle = 1/2\,|\vec{E}(\vec{r})|^2 = 1/2 I(\vec{r})$$

Scattering force: take the imaginary part of polarisability:

$$\vec{F}_{\text{abs+scatt}} = \frac{\left|\vec{E}_0\right|^2}{8\pi}(\sigma_{\text{abs}} + \sigma_{\text{scatt}})\frac{\vec{k}}{k}$$

$$\sigma = \sigma_{\text{abs}} + \sigma_{\text{scatt}} = 4\pi k a^3 \operatorname{Im}\left\{\frac{\varepsilon-1}{\varepsilon+2}\right\} + \frac{8\pi}{3}k^4 a^6 \left|\frac{\varepsilon-1}{\varepsilon+2}\right|^2$$

absorption losses

radiation losses

Dielectric particles: no absorption $\sigma_{\mathrm{abs}} \approx 0$

scattering force: result of the difference between the momentum of the input beam (in the direction of propagation) and the secondary photons scattered by the induced oscillating dipole (in all directions)

$$\left\langle \vec{F}_{\mathrm{scatt}}(\vec{r}) \right\rangle = \frac{\sigma_{\mathrm{scatt}} \left\langle \vec{S}_{\mathrm{P}}(\vec{r},t) \right\rangle}{c/n_{\mathrm{m}}} = \hat{z}\left(n_{\mathrm{m}}/c\right)\sigma_{\mathrm{scatt}} I(\vec{r})$$

plugging expression for radiative scattering losses

$$\left\langle \vec{F}_{\mathrm{scatt}}(\vec{r}) \right\rangle = \hat{z}\,\frac{n_{\mathrm{m}}}{c}\,\frac{8\pi}{3}(ka)^{4}\,a^{2}\left(\frac{m^{2}-1}{m^{2}+2}\right)^{2} I(\vec{r})$$

Gradient force:

(time average)

$$\mathbf{F} = \frac{1}{2}\alpha\nabla E^2 = \frac{2\pi n_0 a^3}{c}\left(\frac{m^2-1}{m^2+2}\right)\nabla I(\mathbf{r})$$

Scattering force:

(conservation of momentum)

$$\mathbf{F}_{\text{scat}}(\mathbf{r}) = \frac{k^4\alpha^2}{6\pi c n_0^3 \epsilon_0^2} I(\mathbf{r})\hat{z} = \frac{8\pi n_0 k^4 a^6}{3c}\left(\frac{m^2-1}{m^2+2}\right)^2 I(\mathbf{r})\hat{z}$$

Balance of these two forces dictates the spatial location of the stable position!

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Maxwell's stress tensor

Continuity equation for electric charge and current density:

$$\frac{\partial \rho(\mathbf{r},t)}{\partial t} + \nabla \cdot \mathbf{j}(\mathbf{r},t) = 0$$

charge = conserved quantity

$$\mathbf{j}(\mathbf{r},t) = \rho(\mathbf{r},t)\mathbf{v}(\mathbf{r},t)$$

charges move along closed lines

similar conservation equation for momentum

What momentum can be transferred from an electromagnetic field to a charge or charge distribution?

goal:

$$\frac{dP_{\mathrm{mech}}}{dt} = \mathbf{F}(t) = \iiint \mathbf{f}(\mathbf{r},t)dV$$

external fields exert force acting on the charge

→ moves to new position

charge acquired momentum that flowed from the fields into the charge

Maxwell Stress Tensor

force per volume acting on free charges and currents in the presence of em fields

Lorentz force $$\boxed{\mathbf{f} = \rho\mathbf{E} + \rho\mathbf{v}\times\mathbf{B} = \rho\mathbf{E} + \mathbf{j}\times\mathbf{B}}$$

goal: express everything in terms of fields

with: $\nabla\cdot\mathbf{D} = \rho$ and $\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \mathbf{j}$

$$\boxed{\mathbf{f} = \mathbf{E}[\nabla\cdot\mathbf{D}] - \mathbf{B}\times\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B}}$$

with: $\frac{\partial}{\partial t}[\mathbf{D}\times\mathbf{B}] = \frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B} + \mathbf{D}\times\frac{\partial\mathbf{B}}{\partial t}$

$$\boxed{\mathbf{f} = \mathbf{E}[\nabla\cdot\mathbf{D}] - \mathbf{B}\times\nabla\times\mathbf{H} + \mathbf{D}\times\frac{\partial\mathbf{B}}{\partial t} - \frac{\partial}{\partial t}[\mathbf{D}\times\mathbf{B}]}$$

with: $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$

$$\boxed{\mathbf{f} = \mathbf{E}[\nabla\cdot\mathbf{D}] - \mathbf{B}\times\nabla\times\mathbf{H} - \mathbf{D}\times\nabla\times\mathbf{E} - \frac{\partial}{\partial t}[\mathbf{D}\times\mathbf{B}]}$$

Inserting a zero $\mathbf{div}\,\mathbf{B}(\mathbf{r},t) = 0$ for a highly symmetric expression

$$\boxed{\mathbf{f} = \mathbf{E}[\nabla\cdot\mathbf{D}] + \mathbf{H}[\nabla\cdot\mathbf{B}] - \mathbf{B}\times\nabla\times\mathbf{H} - \mathbf{D}\times\nabla\times\mathbf{E} - \frac{\partial}{\partial t}[\mathbf{D}\times\mathbf{B}]}$$

derive in the following the Minkowski Stress tensor

historically Maxwell stress tensor but Maxwell
derived it only for vacuum

assuming linear relation between electric
field and displacement field

Simplification (proof in excercise)

$$\mathbf{E}[\boldsymbol{\nabla}\cdot\mathbf{D}] - \mathbf{D}\times\boldsymbol{\nabla}\times\mathbf{E} = \frac{\partial}{\partial x_\beta}\left\{E_\alpha D_\beta - \frac{1}{2}\delta_{\alpha\beta}E_\gamma D_\gamma\right\} = \boldsymbol{\nabla}\cdot\left\{\mathbf{ED} - \frac{1}{2}\mathbf{I}[\mathbf{E}\cdot\mathbf{D}]\right\}$$

tensor product
between ED,
also ⊗

$$\mathbf{H}[\boldsymbol{\nabla}\cdot\mathbf{B}] - \mathbf{B}\times\boldsymbol{\nabla}\times\mathbf{H} = \boldsymbol{\nabla}\cdot\left\{\mathbf{HB} - \frac{1}{2}\mathbf{I}[\mathbf{H}\cdot\mathbf{B}]\right\}$$

$$\boxed{\mathbf{f} = \boldsymbol{\nabla}\cdot\left\{\mathbf{ED} + \mathbf{HB} - \frac{1}{2}\mathbf{I}[\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}]\right\} - \frac{\partial}{\partial t}[\mathbf{D}\times\mathbf{B}]}$$

Since $\mathbf{f} = \frac{\partial \mathbf{p}}{\partial t}$, with $\mathbf{p}(\mathbf{r}, t)$ the momentum density of the free charges we obtain

$$\boxed{\frac{\partial \mathbf{p}}{\partial t} + \frac{\partial}{\partial t}[\mathbf{D}\times\mathbf{B}] = -\boldsymbol{\nabla}\cdot\left\{-\left(\mathbf{ED} + \mathbf{HB} - \frac{1}{2}\mathbf{I}[\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}]\right)\right\}}$$

$$\boxed{\frac{\partial \mathbf{p}}{\partial t} + \frac{\partial}{\partial t}[\mathbf{D}\times\mathbf{B}] = -\boldsymbol{\nabla}\cdot\left\{-\left(\mathbf{ED} + \mathbf{HB} - \frac{1}{2}\mathbf{I}[\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}]\right)\right\}}$$

change of momentum of free charges

change of momentum density of electromagnetic field

negative of the divergence of the momentum current density

momentum stored in the field per volume

$$\mathbf{g}_{\text{Minkowski}} = \mathbf{D}\times\mathbf{B}$$

momentum current density:

$$\mathbf{J} = -\left(\mathbf{ED} + \mathbf{HB} - \frac{1}{2}\mathbf{I}[\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}]\right)$$

Minkowski stress tensor defined as the negative momentum current density:

$$\mathbf{T} = \mathbf{ED} + \mathbf{HB} - \frac{1}{2}\mathbf{I}[\mathbf{E}\cdot\mathbf{D} + \mathbf{H}\cdot\mathbf{B}]$$

vacuum:

$$\mathbf{T} = \varepsilon_0\mathbf{EE} + \frac{1}{\mu_0}\mathbf{BB} - \frac{1}{2}\mathbf{I}\left[\varepsilon_0\mathbf{E}\cdot\mathbf{E} + \frac{1}{\mu_0}\mathbf{B}\cdot\mathbf{B}\right]$$

component notation: $\boxed{T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2}\delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2}\delta_{ij} B^2 \right)}$

Components T_{ij} of the stress tensor have the following meaning:

force per unit area in direction $\mathbf{e}_i$ acting on the surface being normal in direction $\mathbf{e}_j$

$\longrightarrow T_{ii}$ pressures (forces normal to surfaces)

$\longrightarrow T_{ij}$ with $i \neq j$ shears (forces parallel to surfaces)

tensor is symmetric: $T_{ij} = T_{ji}$

equation is a continuity equation

$$\mathbf{p}_{\text{total}} = \mathbf{p}_{\text{mech}} + \mathbf{g}_{\text{Minkowski}}$$

$$\boxed{\frac{\partial \mathbf{p}_{\text{mech}}}{\partial t} + \frac{\partial \mathbf{g}_{\text{Minkowski}}}{\partial t} = \nabla \mathbf{T}_{\text{Minkowski}}}$$

total force:

$$\mathbf{F} = \int_{\mathcal{V}} \mathbf{f} d^3\mathbf{r} = \int_{\mathcal{V}} \left(\nabla \cdot \overleftrightarrow{\mathbf{T}} - \epsilon_0 \mu_0 \frac{\partial \mathbf{S}}{\partial t} \right) d^3\mathbf{r}$$

$$\boxed{= \int_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \epsilon_0 \mu_0 \frac{\partial}{\partial t} \int_{\mathcal{V}} \mathbf{S} d^3\mathbf{r}}$$

Comments:

- momentum of propagating em-wave increases upon entering a dielectric medium proportional to refractive index n (no absorption nor dispersion)
- definition of momentum not obvious
- Abraham: $\mathbf{g}_{\mathrm{Abraham}} = \frac{1}{c^2} \mathbf{E} \times \mathbf{H}$
- momentum reduces in media proportional to n
- different stress tensor in nonlinear media / same in linear media
- Abraham-Minkovski controversy is a splitting problem

reformulating equation above but with the purpose to end up with

$$\mathbf{D} \times \mathbf{B} = \epsilon \mathbf{E} \times \mathbf{B} = \frac{\epsilon\mu}{\mu} \mathbf{E} \times \mathbf{B} = \frac{1}{c^2_{\mathrm{Medium}}} \mathbf{E} \times \mathbf{H} = \frac{1}{c^2} n^2 \mathbf{E} \times \mathbf{H}$$

$$\vec{\nabla}\overset{\leftrightarrow}{T}^{M}=\frac{\partial}{\partial t}\left(\vec{p}_{mech}+\vec{D}\times\vec{B}\right)\;=\frac{\partial}{\partial t}\left(\vec{p}_{mech}+\frac{1}{c^2}n^2\vec{E}\times\vec{H}\right)$$

$$=\frac{\partial}{\partial t}\left(\vec{p}_{mech}+\underbrace{\frac{1}{c^2}\left(n^2-1\right)\vec{E}\times\vec{H}}_{\vec{g}^M-\vec{g}^A}+\underbrace{\frac{1}{c^2}\vec{E}\times\vec{H}}_{\vec{g}^A}\right)$$

$$=\underbrace{\frac{\partial}{\partial t}\vec{p}_{mech}+\vec{f}^A}_{\frac{\partial}{\partial t}\vec{p}^A_{mech}}+\frac{\partial\vec{g}^A}{\partial t}$$

additional Abraham force density assigned to medium : $\vec{f}^A=\frac{\partial\vec{g}^M}{\partial t}-\frac{\partial\vec{g}^A}{\partial t}$

force acting on medium

$$\vec{F}_A\left(t\right)=\int_V\frac{\partial\vec{p}^A_{mech}}{\partial t}\mathrm{d}V=\int_V\vec{\nabla}\overset{\leftrightarrow}{T}^{M}\,\mathrm{d}V-\int_V\frac{\partial\vec{g}^A}{\partial t}\mathrm{d}V=\int_{\partial V}\overset{\leftrightarrow}{T}^{M}\,\mathrm{d}\vec{A}-\int_V\frac{\partial\vec{g}^A}{\partial t}\mathrm{d}V$$

Abraham force tensor:

$$T^A_{ij}=\tfrac{1}{2}\left(E_iD_j+E_jD_i\right)+\tfrac{1}{2}\left(H_iB_j+H_jB_i\right)-\tfrac{1}{2}\left(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B}\right)\delta_{ij}$$

Symmetric: $\overset{\leftrightarrow}{T} = \left(\overset{\leftrightarrow}{T}\right)^T$

linear medium identical to Minkowski: $D_i = \varepsilon E_i \text{ und } B_i = \mu H_i$

$$\begin{aligned} T_{ij}^A &= \frac{1}{2}\left(E_i D_j + E_j D_i\right) + \frac{1}{2}\left(H_i B_j + H_j B_i\right) - \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right)\delta_{ij} \\ &= \frac{1}{2}\left(E_i E_j \varepsilon + E_j E_i \varepsilon\right) + \frac{1}{2}\left(H_i H_j \mu + H_j H_i \mu\right) - \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right)\delta_{ij} \\ &= E_i E_j \varepsilon + H_i H_j \mu - \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right)\delta_{ij} \\ &= E_i D_j + H_i B_j - \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right)\delta_{ij} \\ &= T_{ij}^M \end{aligned}$$

In frequency domain: $\langle \bar{\mathbf{F}}(\boldsymbol{\omega})\rangle = \langle \oint (\bar{\mathbf{T}}(\mathbf{r},\omega)\cdot\mathbf{n})\mathrm{d}A\rangle$

$$= \int_S \left\{ \frac{\varepsilon_0\varepsilon(\omega)}{2}\Re[(\bar{\mathbf{E}}(\mathbf{r},\omega)\cdot\mathbf{n})\bar{\mathbf{E}}^*(\mathbf{r},\omega)] - \frac{\varepsilon_0\varepsilon(\omega)}{4}\left(\bar{\mathbf{E}}(\mathbf{r},\omega)\cdot\bar{\mathbf{E}}^*(\mathbf{r},\omega)\right)\mathbf{n} \right.$$

$$\left. + \frac{\mu_0\mu(\omega)}{2}\Re[(\bar{\mathbf{H}}(\mathbf{r},\omega)\cdot\mathbf{n})\bar{\mathbf{H}}^*(\mathbf{r},\omega)] - \frac{\mu_0\mu(\omega)}{4}\left(\mathbf{H}(\mathbf{r},\omega)\cdot\bar{\mathbf{H}}^*(\mathbf{r},\omega)\right)\mathbf{n} \right\} dl'$$

where dl' is the length of a line segment of the surface.

The net radiation torque on the particle is calculated by

$$\langle \boldsymbol{\tau}(\boldsymbol{\omega})\rangle = \langle \oint \mathbf{r}\times(\bar{\mathbf{T}}(\mathbf{r},\omega)\cdot\mathbf{n})\mathrm{d}A\rangle.$$

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