Theoretical Optics

Diffraction Theory: Scalar approximation

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Scalar approximation

Remember:

Propagation of a pulsed beam

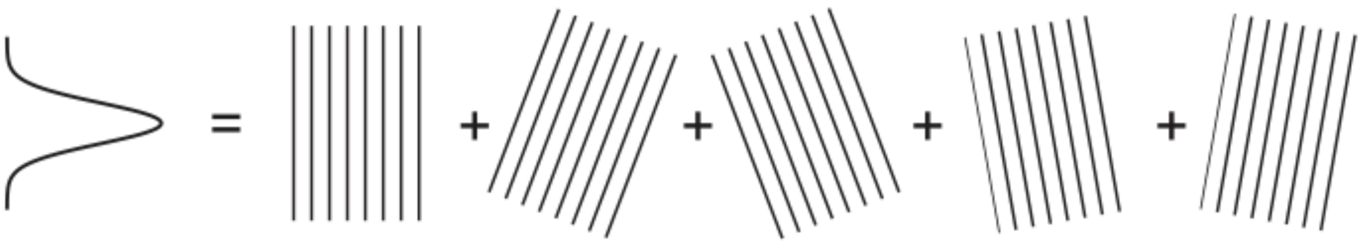
(finite transverse width and finite duration)

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \bar{\mathbf{E}}(\mathbf{k},\omega) e^{i(\mathbf{k}(\omega)\cdot\mathbf{r}-\omega t)} d^3k\, d\omega$$

continuous superposition stationary plane waves
different frequencies and propagation directions

Bundle:
(finite width, time harmonic)

$$\mathbf{E}(\mathbf{r},t) = \int_{-\infty}^{\infty} \bar{\mathbf{E}}(\mathbf{k}) \exp\left[i\left(\mathbf{k}\mathbf{r} - \omega\, t\right)\right] d^3k$$



Assumptions: → transparency region

$\varepsilon(\omega) = \varepsilon'(\omega) > 0$ → normal modes are propagating and evanescent plane waves

Solving Helmholtz equation :

$$\Delta \bar{\mathbf{E}}(\mathbf{r},\omega) + \frac{\omega^2}{c_0^2}\varepsilon(\omega)\bar{\mathbf{E}}(\mathbf{r},\omega) = 0$$

approximating the vector with a single scalar quantity

$$\bar{\mathbf{E}}(\mathbf{r},\omega) \to \bar{E}_y(\mathbf{r},\omega)\mathbf{e}_y \to \bar{E}_y(\mathbf{r},\omega) \to u(\mathbf{r},\omega)$$

- exact for one-dimensional bundles with a linear polarization
- an approximation in the two-dimensional case

$$\Delta u(\mathbf{r},\omega) + \frac{\omega^2}{c_0^2}\varepsilon(\omega)u(\mathbf{r},\omega) = 0$$

$$\Delta u(\mathbf{r},\omega) + k^2(\omega)u(\mathbf{r},\omega) = 0$$

Scalar Helmholtz equation

(we will frequently drop the frequency argument)

Justification

$$\bar{\mathbf{E}}(\mathbf{r},\omega) = \int_{-\infty}^{\infty} \mathbf{e}(\mathbf{k},\omega) e^{i(k_x x + k_y y + k_z z)} dk_x dk_y dk_z$$

$$\boxed{\mathrm{div}\bar{\mathbf{E}}(\mathbf{r},\omega) = 0 \quad \rightarrow \quad k_x e_x + k_y e_y + k_z e_z = 0}$$

a) one-dimensional bundle: invariant in e.g. y-direction

$k_y = 0 \quad \rightarrow \quad e_y \ \rightarrow \ u$ scalar approximation exact

b) two-dimensional bundle finite in x- and y-direction

$k_x, k_y \neq 0$ assume: $k_x = 0 \ \rightarrow \ k_y e_y + k_z e_z = 0$

$$e_z(\mathbf{k},\omega) = -\frac{k_y}{k_z} e_y(\mathbf{k},\omega) = -\frac{k_y}{\sqrt{k^2 - k_x^2 - k_y^2}} e_y(\mathbf{k},\omega)$$

$$\boxed{e_z(\mathbf{k},\omega) \approx -\frac{k_y}{k} e_y(\mathbf{k},\omega) \approx 0}$$ paraxial approximation

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Diffraction Theory: Angular spectrum method

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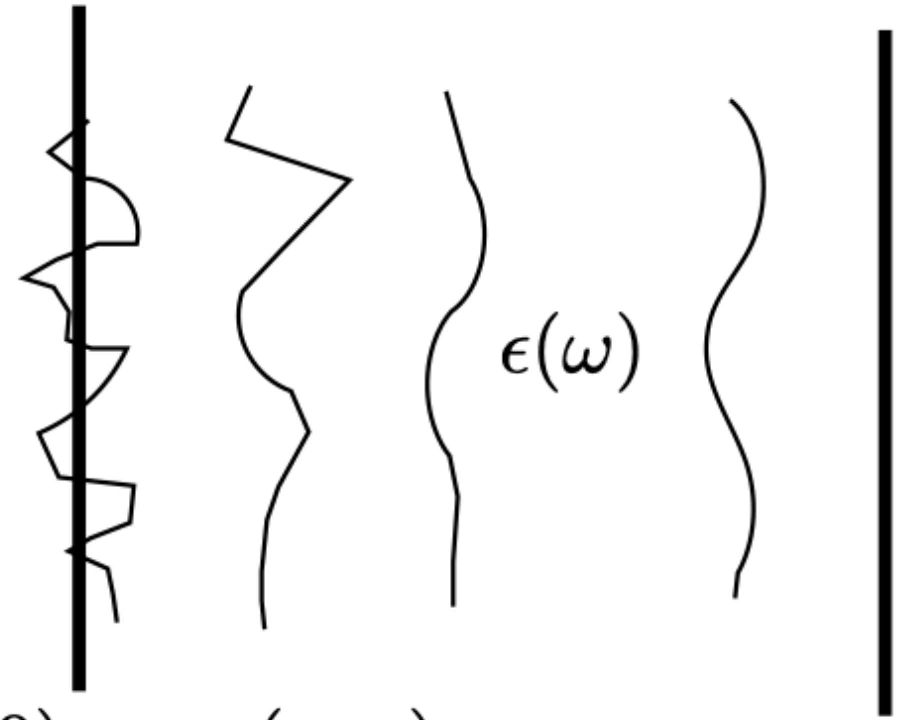
Angular spectrum

- exact solution, good for computer
- find the field in $z > 0$ using the field in an initial plane at, e.g., $z = 0$



$u(x, y, 0) = u_0(x, y)$ $\qquad$ $u(x, y, z)$

$$\Delta u(\mathbf{r},\omega) + k^2(\omega) u(\mathbf{r},\omega) = 0$$

- Fourier transform $u(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} U(\mathbf{k}, \omega) e^{i\mathbf{k}(\omega)\cdot\mathbf{r}}\, d^3k$

- pairs of allowed values == dispersion relation

$$\mathbf{k}^2(\omega) = k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c_0^2}\varepsilon(\omega)$$

- two components necessarily fixes the third
- propagation direction of z -> fix k_x and k_y

change of notation $k_x = \alpha, k_y = \beta, \text{and } k_z = \gamma$

$$u(\mathbf{r}) = \int_{-\infty}^{\infty} U(\alpha, \beta; z) e^{i(\alpha x + \beta y)}\, d\alpha d\beta$$

- α and β are called spatial frequencies
- semi-colon denotes parametrically on z
- this Ansatz into scalar Helmholtz equation

$$\left(\frac{d^2}{dz^2} + k^2(\omega) - \alpha^2 - \beta^2\right) U(\alpha, \beta; z) = 0$$

$$\left(\frac{d^2}{dz^2} + \gamma^2(\alpha, \beta, \omega)\right) U(\alpha, \beta; z) = 0$$

solution $\boxed{U(\alpha, \beta; z) = U_1(\alpha, \beta) e^{i\gamma(\alpha,\beta)z} + U_2(\alpha, \beta) e^{-i\gamma(\alpha,\beta)z}}$

Two types of solution:

a) $\gamma^2(\omega) > 0 \rightarrow \alpha^2 + \beta^2 \leq k^2(\omega)$ homogenous propagating waves

b) $\gamma^2(\omega) < 0 \rightarrow \alpha^2 + \beta^2 > k^2(\omega)$ evanescent waves

$\sim e^{-i\gamma(\omega)z}$ unphysical since it may exponentially grow

set $\longrightarrow$ $U_2(\alpha, \beta) = 0$ $\boxed{U(\alpha, \beta; z) = U_1(\alpha, \beta) e^{i\gamma(\alpha,\beta)z}}$

initial value condition

$$U(\alpha, \beta; z) = U_1(\alpha, \beta; 0)e^{i\gamma(\alpha,\beta)z}$$

$$U(\alpha, \beta; z) = U_0(\alpha, \beta)e^{i\gamma(\alpha,\beta)z}$$

total field in relevant half space

$$\boxed{u(\mathbf{r}) = \iint_{-\infty}^{\infty} U_0(\alpha, \beta)e^{i\gamma(\alpha,\beta)z}e^{i(\alpha x+\beta y)}d\alpha d\beta}$$

$$U_0(\alpha, \beta) = \left(\frac{1}{2\pi}\right)^2 \iint_{-\infty}^{\infty} u_0(x, y)e^{-i(\alpha x+\beta y)}dxdy$$

the "angular spectrum" (→ U_0) the "angular frequencies" (→ α, β)

$$U(\alpha, \beta; z) = U_0(\alpha, \beta)e^{i\gamma(\alpha,\beta)z}$$

transfer function of free space (→ $e^{i\gamma(\alpha,\beta)z}$)

Cause for diffraction: different phase accumulation in the propagation direction of the different spatial frequency components characterised by α and β.

Diffraction as an example of a linear response theory

Fourier space **Real space**

linear equation

$$\left(\frac{d^2}{dz^2}+\gamma^2(\omega)\right)U(\alpha,\beta;z)=0$$

$$\Delta u(\mathbf{r})+k^2u(\mathbf{r})=0$$

solution

$$U(\alpha,\beta;z)=H(\alpha,\beta;z)U_0(\alpha,\beta)$$

action cause

$$u(x,y,z)=\iint_{-\infty}^{\infty}h(x-x',y-y';z)u_0(x',y')dx'dy'$$

action cause

transfer function

$$H(\alpha,\beta;z)=\exp\left[\mathbf{i}\gamma(\alpha,\beta)z\right]$$

$$\gamma(\alpha,\beta)=\sqrt{k^2-\alpha^2-\beta^2}$$

response function

$$h(x,y;z)=\left(\frac{1}{2\pi}\right)^2\iint_{-\infty}^{\infty}H(\alpha,\beta;z)\exp\left[\mathbf{i}(\alpha x+\beta y)\right]d\alpha d\beta$$

angular spectrum algorithm

$$u_0(x,y)\xrightarrow{\mathbf{FT}^{-1}}U_0(\alpha,\beta)\;\rightarrow\;U(\alpha,\beta;z)=H(\alpha,\beta;z)U_0(\alpha,\beta)\;\xrightarrow{\mathbf{FT}}\;u(x,y,z)$$

Fresnel approximation

!in Fourier space!

$$U_0(\alpha, \beta) \neq 0 \quad \text{for} \quad \alpha^2 + \beta^2 \ll k^2$$

plane waves that form the beams:
only small inclination relative to the principle propagation direction
(=paraxial approximation)

$$\gamma = \sqrt{k^2 - \alpha^2 - \beta^2} \longrightarrow \gamma = \sqrt{k^2 - \alpha^2 - \beta^2} \approx k\left(1 - \frac{\alpha^2 + \beta^2}{2k^2}\right) = k - \frac{\alpha^2 + \beta^2}{2k}$$

transfer function in Fresnel approximation

$$e^{i\gamma(\alpha,\beta)z} \approx e^{ik_0 n z} e^{-i\frac{z}{2k_0 n}(\alpha^2+\beta^2)}$$

$$u(\mathbf{r}) = \iint_{-\infty}^{\infty} U_0(\alpha, \beta) e^{i\gamma(\alpha,\beta)z} e^{i(\alpha x + \beta y)} d\alpha d\beta \longrightarrow \text{Fresnel}$$

$$u_{\text{Fresnel}}(x, y, z_B) = \iint_{-\infty}^{\infty} U_+(\alpha, \beta; z_A) e^{ik_0 n z_B} e^{-i\frac{z_B}{2k_0 n}(\alpha^2+\beta^2)} e^{i(\alpha x + \beta y)} d\alpha d\beta$$

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