Theoretical Optics

Diffraction Theory: Helmholtz and Kirchhoff - intro

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Integral Theorem of Helmholtz and Kirchhoff

Helmholtz and Kirchhoff → Rayleigh-Sommerfeld → Fresnel → Fraunhofer

Strategy

- express the field components in their values at $z = 0$ by means of only one surface integral
- using the integral theorem of Helmholtz and Kirchhoff
- expresses the field at an arbitrary point in space in terms of an integral across a surface surrounding the space of interest
- specific choice of the Green's function this reduces to the Rayleigh-Sommerfeld diffraction
- approximating the Green's function leads to Fresnel and even further to Fraunhofer diffraction

- $u(\mathbf{r})$ and $G(\mathbf{r})$ complex functions with single-valued and continuous first and second derivatives.
- V denotes a volume that is bounded by the closed surface S.
- Green's second identity says

$$\iiint_V \left(u(\mathbf{r}')\Delta G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\Delta u(\mathbf{r}')\right) d^3r'$$

$$= \iint_S \left(u(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} - G(\mathbf{r},\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}}\right) d^2r'$$

- derivative is done with respect to the outward normal direction.
- Require $u(\mathbf{r})$ solution to Helmholtz Eqn. for homogenous isotropic medium

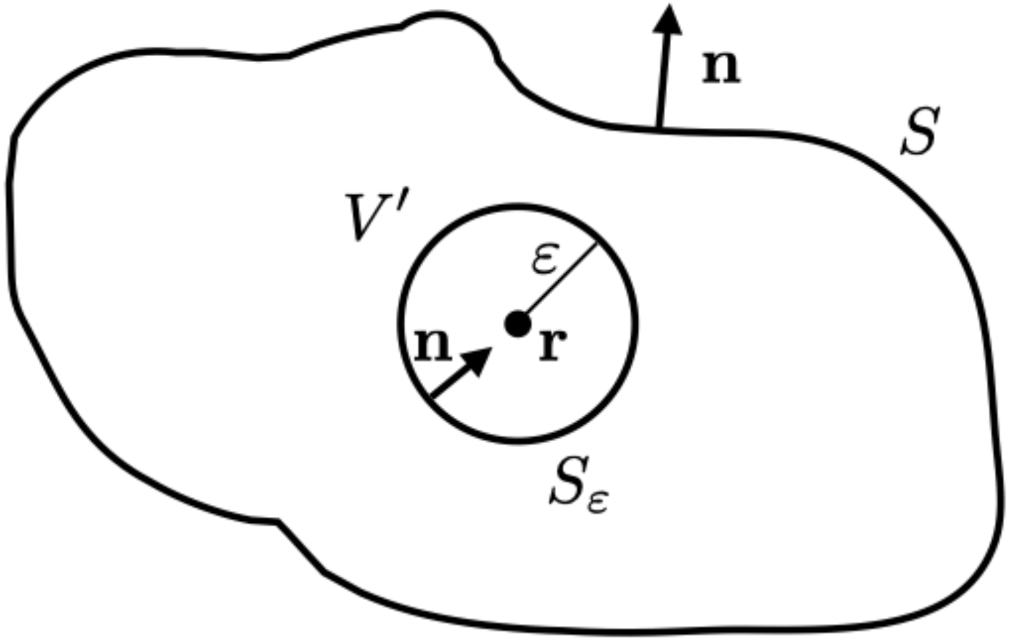
$$(\Delta + k_0^2 n^2)u(\mathbf{r}) = 0$$

- Green's function that solves the problem

$$(\Delta + k_0^2 n^2)G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}' - \mathbf{r})$$

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}' - \mathbf{r}|}$$

Specific situation



- excludes a small sphere of radius ϵ around the point $\mathbf{r}$ that is the point of our observation to avoid the singularity of the Greens function
- consider limit of this vanishing volume going to zero
- This allows to write $(\Delta + k_0^2 n^2) G(\mathbf{r}, \mathbf{r}') = 0$
- lhs of Green's second identify vanishes

final result:

$$\iint_S \left(u(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial \mathbf{n}} - G(\mathbf{r}, \mathbf{r}') \frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} \right) d^2 r' = 0$$

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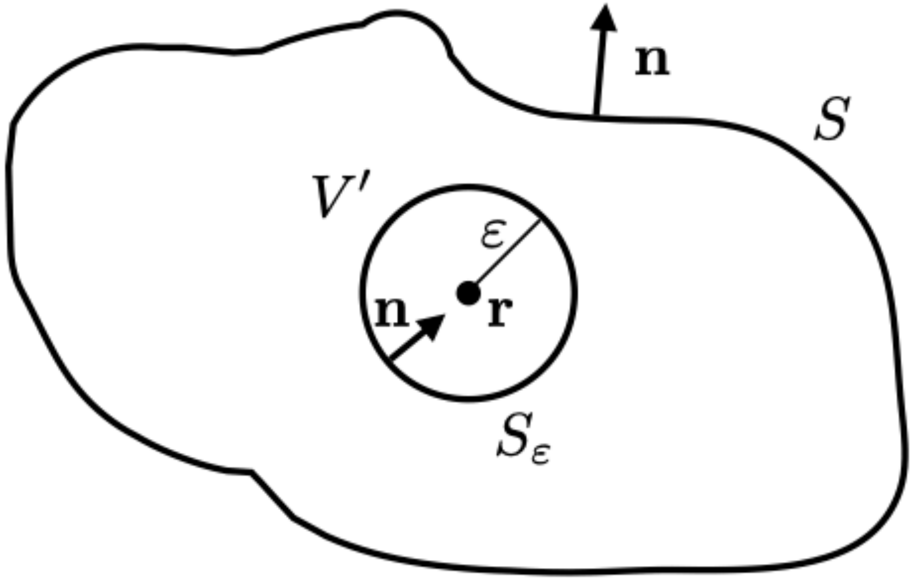
Considered geometry:



Known Green's function:

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}$$

Left over from Green's second identity:

$$\iint_S \left(u(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} - G(\mathbf{r},\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}}\right) d^2r' = 0$$

Separating integral in 2 surfaces for inner and outer boundary:

$$-\iint_{S_\epsilon} \left(u(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} - G(\mathbf{r},\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}}\right) d^2r' = \iint_S \left(u(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} - G(\mathbf{r},\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}}\right) d^2r'$$

Evaluating the left and the right hand side individually

Outer surface (right hand side):

not much simplification except analytical solution to derivative of Green's function

$$\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} = \cos\big(\angle(\mathbf{n},\mathbf{r}'-\mathbf{r})\big)\left(ik_0 n - \frac{1}{|\mathbf{r}'-\mathbf{r}|}\right)\frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}$$

angle between the outward normal and the vector $\mathbf{r}' - \mathbf{r}$

Inner surface (left hand side):

spherical shape for the surface S_ϵ and a constant radius $\epsilon = |\mathbf{r}' - \mathbf{r}|$

specifies Green's function to: $G(\mathbf{r},\mathbf{r}') = \dfrac{e^{ik_0 n\epsilon}}{\epsilon}$

angle between surface normal and radial coordinate: $\cos\big(\angle(\mathbf{n},\mathbf{r}'-\mathbf{r})\big) = -1$

two vectors are anti-parallel as the surface norm point outwards combining all these ingredients:

$$\iint_{S_\epsilon}\left(u(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} - G(\mathbf{r},\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}}\right)d^2r' = \iint_{S_\epsilon}\left(u(\mathbf{r}')\left(\frac{1}{\epsilon} - ik_0 n\right)\frac{e^{ik_0 n\epsilon}}{\epsilon} - \frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}}\frac{e^{ik_0 n\epsilon}}{\epsilon}\right)d^2r'$$

- choosing ϵ increasingly small
- applying mean value theorem for integration
- the integrals become the area of a sphere × $u(\mathbf{r})$ and × $\frac{\partial u(\mathbf{r})}{\partial \mathbf{n}}$

$$\iint_{S_\epsilon}\left(u(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} - G(\mathbf{r},\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}}\right)d^2r' \simeq 4\pi\epsilon^2\left(u(\mathbf{r})\left(\frac{1}{\epsilon} - ik_0 n\right)\frac{e^{ik_0 n\epsilon}}{\epsilon} - \frac{\partial u(\mathbf{r})}{\partial \mathbf{n}}\frac{e^{ik_0 n\epsilon}}{\epsilon}\right)$$

only the first term survives in the limiting case of $\epsilon \to 0$

final result for inner surface (left hand side)

$$\iint_{S_\epsilon}\left(u(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} - G(\mathbf{r},\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}}\right)d^2r' = 4\pi u(\mathbf{r})$$

final result:

$$u(\mathbf{r}) = \frac{1}{4\pi}\iint_{S}\left(\frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} - u(\mathbf{r}')\frac{\partial}{\partial \mathbf{n}}\frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\right)d^2r'$$

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Diffraction Theory: Kirchhoff formulation

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Helmholtz and Kirchhoff formulation

$$u(\mathbf{r}) = \frac{1}{4\pi}\iint_S \left(\frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} - u(\mathbf{r}')\frac{\partial}{\partial \mathbf{n}}\frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\right) d^2 r'$$

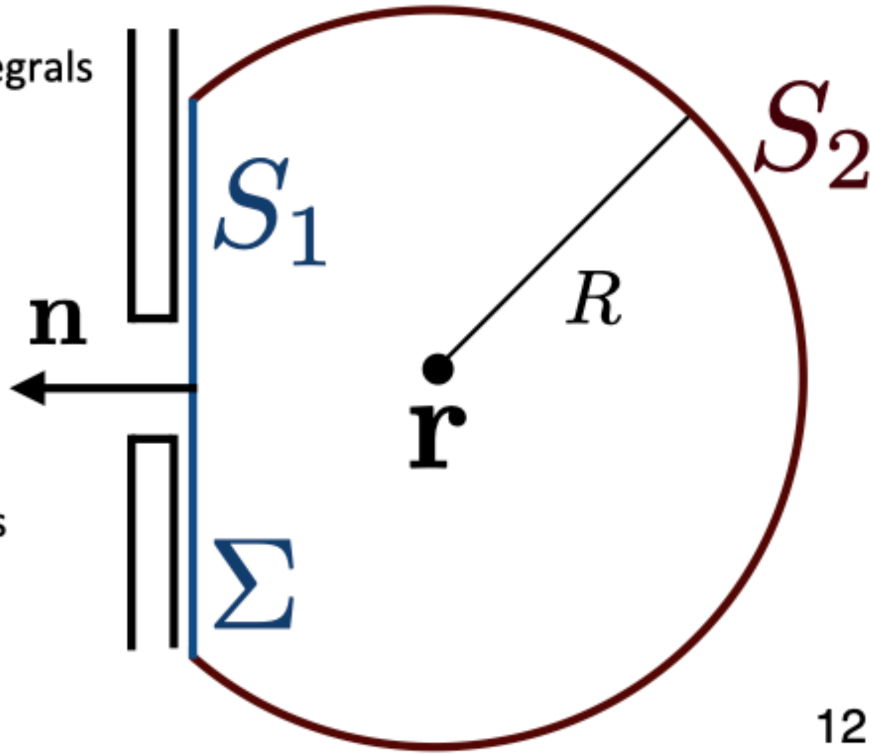
Kirchhoff formulation

connection between angular spectrum and Helmholtz and Kirchhoff formulation

requires information on the field in an initial plane

requires information on the field and its derivative on a surface enclosing the point of interest

- Separating the solution above to a sum of two integrals
- First is integration across the screen
- Second is integration across a half sphere
- consider the limit of $R \to \infty$
- Surface area of sphere is in the order of R^2
  - in the integrand we can neglect contributions that decay faster than R^{-2}



modulus of the Green's function $|G(\mathbf{r},\mathbf{r}')| = \left|\frac{e^{ik_0 n|\mathbf{r}\prime-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\right| = \left|\frac{1}{|\mathbf{r}'-\mathbf{r}|}\right| = \frac{1}{R}$

decays as 1/R

assume that the field $u(\mathbf{r})$ equally decays comparable to a spherical wave

Taylor expansion of the outward normal derivative of the Green's function and only retain the term that decays as R^{-1}

$$\frac{\partial}{\partial \mathbf{n}}\frac{e^{ik_0 n|\mathbf{r}\prime-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} = \left(ik_0 n - \frac{1}{R}\right)\frac{e^{ik_0 nR}}{R} = ik_0 nG(\mathbf{r},\mathbf{r}') + \mathcal{O}(R^{-2}) \overset{R\to\infty}{\cong} ik_0 nG(\mathbf{r},\mathbf{r}')$$

integral for only the semi-spherical surface

$$u(\mathbf{r}) = \frac{1}{4\pi}\iint_{S_2}\left(G(\mathbf{r},\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} - u(\mathbf{r}')ik_0 nG(\mathbf{r},\mathbf{r}')\right)d^2r'$$

expressing the integral in terms of solid angles

$$u(\mathbf{r}) = \frac{1}{4\pi}\iint_{S_2} d\Omega G(\mathbf{r},\mathbf{r}')\left(\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} - ik_0 nu(\mathbf{r}')\right)R^2$$

take the limit for $R \to \infty$

$$\xrightarrow{R\to\infty} u(\mathbf{r}) = \frac{1}{4\pi} \iint_{S_2} d\Omega\, R \left(\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} - i k_0 n u(\mathbf{r}') \right)$$

the integral vanishes if we require that

$$\lim_{R\to\infty} R \left(\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} - i k_0 n u(\mathbf{r}') \right) = 0$$

Sommerfeld radiation condition

"The sources must be sources, not sinks of energy. The energy which is radiated from the sources must scatter to infinity; no energy may be radiated from infinity into ... the field."

- this condition is not satisfied by every field but we would like to enforce it

the integral collapses to the integral across the aperture plane

$$u(\mathbf{r}) = \frac{1}{4\pi} \iint_{S_1} \left(\frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} \frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} - u(\mathbf{r}') \frac{\partial}{\partial \mathbf{n}} \frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} \right) d^2 r'$$

Kirchhoff assumptions:

→ restricting to an integral across an aperture instead of an infinite plane

- field behind the aperture is the incident field in the absence of the screen
- outside the aperture the field and its derivative are zero

Thin element approximation:

$$u(\mathbf{r}) = \frac{1}{4\pi} \iint_{\Sigma} \left(\frac{e^{ik_0 n|\mathbf{r}' - \mathbf{r}|}}{|\mathbf{r}' - \mathbf{r}|} \frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} - u(\mathbf{r}') \frac{\partial}{\partial \mathbf{n}} \frac{e^{ik_0 n|\mathbf{r}' - \mathbf{r}|}}{|\mathbf{r}' - \mathbf{r}|} \right) d^2 r'$$

Or in a more general term leaving the choice of the Green's function open

final result:

$$u(\mathbf{r}) = \frac{1}{4\pi} \iint_{\Sigma} \left(G(\mathbf{r}, \mathbf{r}') \frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} - u(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial \mathbf{n}} \right) d^2 r'$$

Remember the requirements:

- scalar theory
- $u(\mathbf{r})$ and $G(\mathbf{r}, \mathbf{r}')$ satisfy the homogeneous wave scalar wave equation
- the satisfaction of the Sommerfeld radiation condition

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