Theoretical Optics

Diffraction Theory: Rayleigh-Sommerfeld

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Rayleigh-Sommerfeld

Final result from Kirchhoff: $$u(\mathbf{r}) = \frac{1}{4\pi}\iint_\Sigma \left(G(\mathbf{r},\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}} - u(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} \right) d^2r'$$

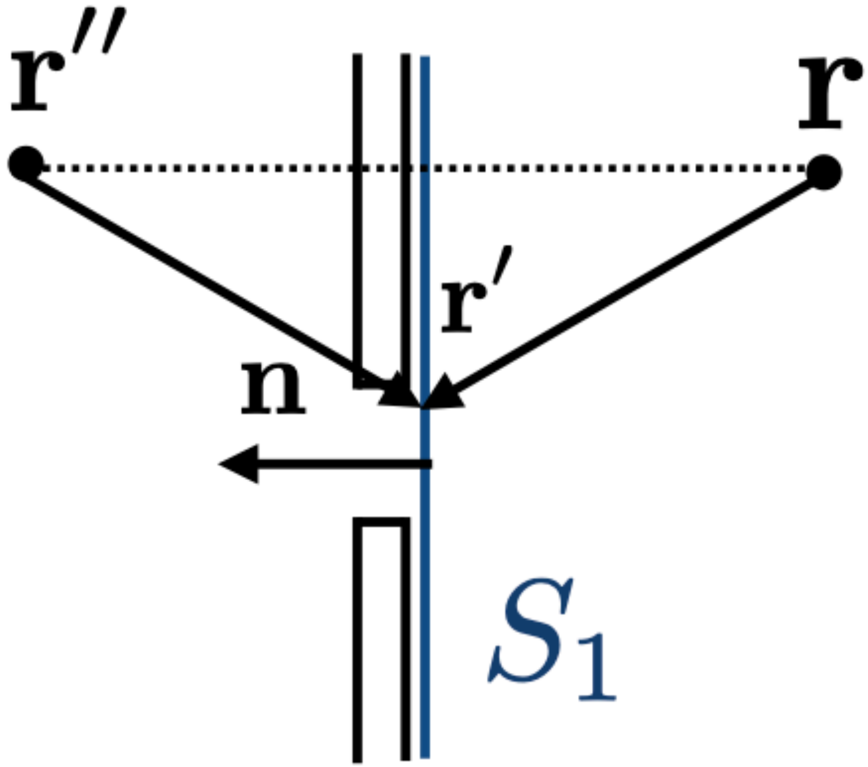
- Kirchhoff has mathematical inconsistencies
- both the field and its normal derivative vanish on the boundary of the screen
- if a two-dimensional potential function and its normal derivative vanish together along any finite curve segment, then that potential function must vanish over the entire plane
- How to mitigate this problem?
  - modify the Green's function such that the development leading to the above equation remains valid, but in addition either $G(\mathbf{r},\mathbf{r}')$ or $\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}}$ shall vanish across the entire surface S_1 $\longrightarrow$ impose different boundary conditions when solving for the Green's function of the half space

→ Dirichlet or Neumann boundary condition

- this was done by Sommerfeld

vanishing field or vanishing normal derivative of the field

- whether the first or the second condition mentioned is fulfilled, we will approach two different Rayleigh-Sommerfeld formulations



- Green's function is not just generated from a point in $\mathbf{r}$
- we add a second Green's function that is located in the same x and y coordinate but which emerges from a spatial point at $-z$.

$$G_{\pm}(\mathbf{r},\mathbf{r}') = \frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|} \pm \frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}''|}}{|\mathbf{r}'-\mathbf{r}''|}$$

with $\mathbf{r}'' = (x, y, -z)$

- Green's functions solutions to same differential equations
- Green's functions have the useful properties that

$$G_{-}(\mathbf{r},\mathbf{r}') = 0 \qquad \frac{\partial G_{+}(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} = 0 \qquad \text{for } \mathbf{r}' \in \mathbf{S}_1$$

requires to know either the field or the normal derivative across the aperture

First Rayleigh-Sommerfeld

$$u_1(\mathbf{r}) = -\frac{1}{4\pi}\iint_\Sigma u(\mathbf{r}')\frac{\partial G_-(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}}d^2r'$$

specify the solution by calculating the normal derivative of $G_-(\mathbf{r},\mathbf{r}')$

$$\frac{\partial G_-(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} = \cos\big(\angle(\mathbf{n},\mathbf{r}'-\mathbf{r})\big)\left(ik_0n - \frac{1}{|\mathbf{r}'-\mathbf{r}|}\right)\frac{e^{ik_0n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}$$

$$-\cos\big(\angle(\mathbf{n},\mathbf{r}'-\mathbf{r}'')\big)\left(ik_0n - \frac{1}{|\mathbf{r}'-\mathbf{r}''|}\right)\frac{e^{ik_0n|\mathbf{r}'-\mathbf{r}''|}}{|\mathbf{r}'-\mathbf{r}''|}$$

- with $|\mathbf{r}'-\mathbf{r}| = |\mathbf{r}'-\mathbf{r}''|$ and $\cos\big(\angle(\mathbf{n},\mathbf{r}'-\mathbf{r})\big) = -\cos\big(\angle(\mathbf{n},\mathbf{r}'-\mathbf{r}'')\big)$

$$\frac{\partial G_-(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} = 2\cos\big(\angle(\mathbf{n},\mathbf{r}'-\mathbf{r})\big)\left(ik_0n - \frac{1}{|\mathbf{r}'-\mathbf{r}|}\right)\frac{e^{ik_0n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}$$

- twice the normal derivative of the Green's function $G(\mathbf{r},\mathbf{r}')$ used in the Kirchhoff formula.

$$u_1(\mathbf{r}) = -\frac{1}{2\pi}\iint_\Sigma u(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}}d^2r'$$

- approximation: the distance of the point of interest relative to the screen is much larger than the wavelength ($|\mathbf{r}'-\mathbf{r}| \gg \lambda$)

$$\frac{\partial G_-(\mathbf{r},\mathbf{r}')}{\partial \mathbf{n}} = 2ik_0 n\cos\big(\angle(\mathbf{n},\mathbf{r}'-\mathbf{r})\big)\frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}$$

final result:

$$u_1(\mathbf{r}) = \frac{n}{i\lambda_0}\iint_\Sigma u(\mathbf{r}')\frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\cos\big(\angle(\mathbf{n},\mathbf{r}'-\mathbf{r})\big)\,d^2r'$$

- same procedure for the different choice of the Green's function

Second Rayleigh-Sommerfeld

$$u_2(\mathbf{r}) = \frac{1}{4\pi}\iint_\Sigma \left(G_+(\mathbf{r},\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}}\right)d^2r'$$

$$u_2(\mathbf{r}) = \frac{1}{2\pi}\iint_\Sigma \left(G(\mathbf{r},\mathbf{r}')\frac{\partial u(\mathbf{r}')}{\partial \mathbf{n}}\right)d^2r'$$

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Diffraction Theory: Fresnel approximation

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Fresnel approximation

final result Rayleigh-Sommerfeld I: $$u_1(\mathbf{r}) = \frac{n}{i\lambda_0}\iint_\Sigma u(\mathbf{r}')\frac{e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\cos(\angle(\mathbf{n},\mathbf{r}'-\mathbf{r}))\,d^2r'$$

- assume that $x', y' \ll z$ and $x, y \ll z$

first implication $\longrightarrow$ $$\frac{1}{|\mathbf{r}'-\mathbf{r}|} \approx \frac{1}{z}$$

second implication $\longrightarrow$ $$\cos(\angle(\mathbf{n},\mathbf{r}'-\mathbf{r})) \approx 1$$

- how to approximate $e^{ik_0 n|\mathbf{r}'-\mathbf{r}|}$

- assume $z' = 0$ $\longrightarrow$ $|\mathbf{r}'-\mathbf{r}|^2 = z^2 + (x'-x)^2 + (y'-y)^2$

making a Taylor series of second order

$$k_0 n|\mathbf{r}' - \mathbf{r}| = k_0 n z \sqrt{1 + \left(\frac{x' - x}{z}\right)^2 + \left(\frac{y' - y}{z}\right)^2} \cong k_0 n z \left[1 + \frac{1}{2}\left(\frac{x' - x}{z}\right)^2 + \frac{1}{2}\left(\frac{y' - y}{z}\right)^2\right]$$

- Plugging this into the first Rayleigh-Sommerfeld diffraction formula leads to

final result:

$$u_{\mathrm{Fresnel}}(\mathbf{r}) = \frac{n e^{i k_0 n z}}{i \lambda_0 z} \iint_{\Sigma} u(\mathbf{r}') e^{i \frac{k_0 n}{2z}\left[(x' - x)^2 + (y' - y)^2\right]} d^2 r'$$

Fresnel as FFT (easy to compute on a computer)

$$u_{\mathrm{Fresnel}}(\mathbf{r}) = \frac{n e^{i k_0 n z} e^{i \frac{k_0 n}{2} \frac{(x^2 + y^2)}{z}}}{i \lambda_0 z} \iint_{\Sigma} \left[u(\mathbf{r}') e^{i \frac{k_0 n}{2} \frac{(x'^2 + y'^2)}{z}}\right] e^{-2\pi i \left(\frac{nx}{\lambda_0 z} x' + \frac{ny}{\lambda_0 z} y'\right)} d^2 r'$$

Spatial frequencies: $\frac{nx}{\lambda_0 z}$ and $\frac{ny}{\lambda_0 z}$

Validity: $n \frac{\left(x'^2 + y'^2\right)\left(x x' + y y'\right)}{\lambda r^3} \ll 1$

$\longrightarrow \frac{n(D/2)^4}{\lambda z_0^3} = \frac{n D^4}{16 \lambda z_0^3} = \frac{\lambda}{n z_0} F^2 \ll 1$ Fresnel number: $F = n(D/2)^2/(\lambda z_0)$

Diffraction Theory: Fresnel approximation

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Diffraction Theory: Fraunhofer approximation

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Fraunhofer approximation

final result Fresnel approximation

$$u_{\text{Fresnel}}(\mathbf{r}) = \frac{n e^{ik_0 nz} e^{i\frac{k_0 n}{2}\frac{(x^2+y^2)}{z}}}{i\lambda_0 z} \iint_\Sigma \left[u(\mathbf{r}') e^{i\frac{k_0 n}{2}\frac{(x'^2+y'^2)}{z}} \right] e^{-2\pi i\left(\frac{nx}{\lambda_0 z}x' + \frac{ny}{\lambda_0 z}y'\right)} d^2 r'$$

approximation: $k_0 n|\mathbf{r}' - \mathbf{r}| \cong k_0 nz + \frac{k_0 n}{2}\frac{(x^2+y^2)}{z} - k_0 n \frac{(xx'+yy')}{z} + \frac{k_0 n}{2}\frac{(x'^2+y'^2)}{z}$

- the last possible approximation is the far-field approximation
- neglecting the terms that depend on x'^2 and y'^2

$$\rightarrow \qquad k_0 n|\mathbf{r}' - \mathbf{r}| \cong k_0 nz + \frac{k_0 n}{2}\frac{(x^2+y^2)}{z} - k_0 n \frac{(xx'+yy')}{z}$$

$$u_{\text{Fraunhofer}}(\mathbf{r}) = \frac{n e^{ik_0 nz + i\frac{k_0 n}{2}\frac{(x^2+y^2)}{z}}}{i\lambda_0 z} \iint_\Sigma u(\mathbf{r}') e^{-2\pi i\left(\frac{nx}{\lambda_0 z}x' + \frac{ny}{\lambda_0 z}y'\right)} d^2 r'$$

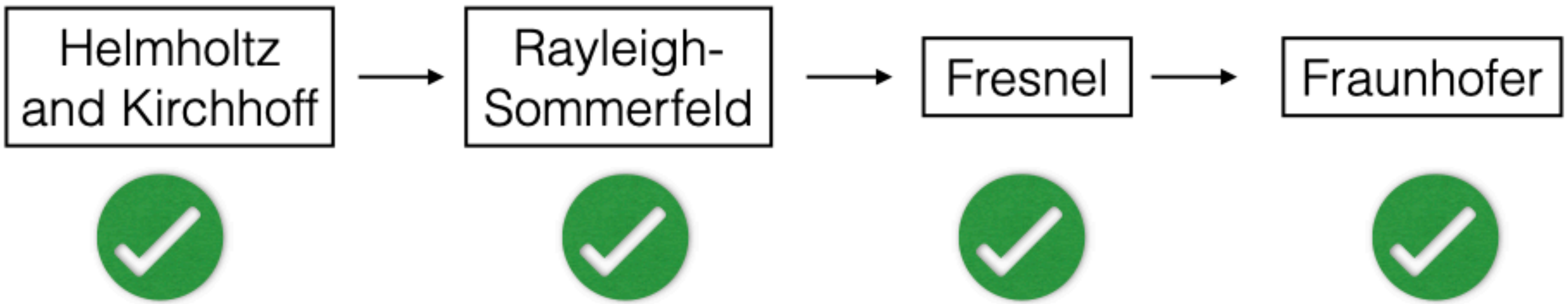
- now this is just the Fourier transform using

$$U\left(k\frac{x}{z},k\frac{y}{z}\right)=\left(\frac{1}{2\pi}\right)^2\iint_\Sigma u(\mathbf{r}')e^{-i\left(k\frac{x}{z}x'+k\frac{y}{z}y'\right)}d^2r'$$

final result:

$$u_{\text{Fraunhofer}}(\mathbf{r})=\frac{n(2\pi)^2}{i\lambda_0 z}e^{ik_0nz+i\frac{k_0n}{2}\frac{(x^2+y^2)}{z}}U\left(k\frac{x}{z},k\frac{y}{z}\right)$$

Validity requirement: $\pi n\dfrac{x'^2+y'^2}{\lambda r}\ll\pi$ $\qquad \dfrac{nD^2}{4\lambda z_0}=F\ll 1$



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Diffraction Theory: Fraunhofer approximation

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Diffraction Theory: Method of stationary points

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Method of stationary points

general method to calculate integrals $I = \iint g(p,q) e^{i\kappa f(p,q)} dp\,dq$

approximately when $\kappa \gg 1$ and $g(p,q)$ varies slowly

here: Fresnel ⟶ Fraunhofer

$\kappa \gg 1$ ⟶ integral oscillates rapidly ⟶ contribution cancel

⟶ contribution only at stationary points p_m and q_m

stationary points defined as

$$\left.\frac{\partial f}{\partial p}\right|_{p_m,q_m} = f_p\big|_{p_m,q_m} = 0$$

$$\left.\frac{\partial f}{\partial q}\right|_{p_m,q_m} = f_q\big|_{p_m,q_m} = 0$$

integral gets sum

$$I = \frac{2\pi}{i\kappa} \sum_{m=1}^{M} \frac{1}{\sqrt{f_{pp(m)} f_{qq(m)} - \frac{1}{4} f_{pq(m)}^2}} g(p_m, q_m) e^{i\kappa f(p_m, q_m)}$$

proof in exercise

application to Fresnel integral

$$u_{\text{Fresnel}}(x, y, z_B) = \iint_{-\infty}^{\infty} U_+(\alpha, \beta; z_A) e^{ik_0 n z_B} e^{-i\frac{z_B}{2k_0 n}(\alpha^2+\beta^2)} e^{i(\alpha x + \beta y)} d\alpha d\beta$$

extending the exponential argument with $k_0 n z_B$

$$u_{\text{Fresnel}}(x, y, z_B) = e^{ik_0 n z_B} \iint_{-\infty}^{\infty} U_+(\alpha, \beta; z_A) e^{ik_0 n z_B \left[\left(\frac{\alpha}{k_0 n} \frac{x}{z_B} + \frac{\beta}{k_0 n} \frac{y}{z_B}\right) - \frac{1}{2}\left(\frac{\alpha^2}{k_0^2 n^2} + \frac{\beta^2}{k_0^2 n^2}\right)\right]} d\alpha d\beta$$

substitution: $p = \frac{\alpha}{k_0 n}$ $q = \frac{\beta}{k_0 n}$ $\kappa = k_0 n z_B$

require that $\kappa \gg 1$

identify

$$f(p,q) = p\frac{x}{z_B} + q\frac{y}{z_B} - \frac{1}{2}(p^2 + q^2)$$

$$u_{\text{Fresnel}}(x,y,z_B) = k_0^2 n^2 e^{i\kappa} \iint_{-\infty}^{\infty} U_+(k_0 np, k_0 nq; z_A) e^{i\kappa f(p,q)} dpdq$$

quantities above

$$\frac{\partial f}{\partial p} = \frac{x}{z_B} - p, \frac{\partial f}{\partial q} = \frac{y}{z_B} - q, \frac{\partial^2 f}{\partial p^2} = \frac{\partial^2 f}{\partial q^2} = -1,\ \frac{\partial^2 f}{\partial p \partial q} = 0$$

1 stationary point only

$$p_1 = \frac{x}{z_B} \text{ and } q_1 = \frac{y}{z_B}$$

$$f(p_1, q_1) = \frac{x^2 + y^2}{2z_B}$$

plug this into the discrete sum

leads to Fraunhofer approximation

$$u_{\mathrm{Fraunhofer}}(x, y, z_B) = \frac{n(2\pi)^2}{i\lambda_0 z_B} e^{ik_0 n z_B} U_+\left(k\frac{x}{z_B}, k\frac{y}{z_B}; z_A\right) e^{i\frac{k_0 n}{2z_B}(x^2+y^2)}$$

at any spatial position only one spatial frequency contributes

$$U_+\left(k\frac{x}{z_B}, k\frac{y}{z_B}; z_A\right)$$

to enforce approximation: $\kappa \gg 1$

- aperture size to wavelength is small
- aperture size to distance small

$$N_F = \frac{a}{\lambda}\frac{a}{z_B} \lesssim 0.1$$

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