Optics in anisotropic media:
derivation of dispersion relation (a)

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Approach 3: Derivation of the dispersion relation: brut force

study monochromatic plane wave $\sim e^{i(\mathbf{k}(\omega)\cdot\mathbf{r}-\omega t)}$

→ but wave number explicitly depends on the propagation direction

→ $k(\omega, \mathrm{direction})$

→ (educated guess): polarization of the normal modes no longer elliptically

→ introduce notation for wave vector as

$$\mathbf{k} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = k \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{with} \quad u_1^2 + u_2^2 + u_3^2 = 1$$

→ goal: find $\omega = \omega(k, u_1, u_2, u_3)$ or $k = k(\omega, u_1, u_2, u_3)$

→ Maxwell's equations in spatial Fourier domain with the ansatz from above

$$\mathbf{k}\cdot\mathbf{D} = 0 \qquad \mathbf{k}\times\mathbf{E} = \omega\mu_0\mathbf{H}$$

$$\mathbf{k}\cdot\mathbf{H} = 0 \qquad \mathbf{k}\times\mathbf{H} = -\omega\mathbf{D}$$

→ wave equation

$$-[\mathbf{k}\times(\mathbf{k}\times\mathbf{E})] = \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon_0}\mathbf{D}$$

$$-\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) + \mathbf{k}^2\mathbf{E} = \frac{\omega^2}{c_0^2}\frac{1}{\varepsilon_0}\mathbf{D}$$

→ in principal coordinate system $D_i = \varepsilon_0\varepsilon_i E_i$

→ in component notation

$$-k_i\sum_{j=1}^{3} k_j E_j + k^2 E_i = \frac{\omega^2}{c_0^2}\varepsilon_i E_i$$

$$\left(\frac{\omega^2}{c_0^2}\varepsilon_i - k^2\right)E_i = -k_i\sum_{j=1}^{3} k_j E_j$$

→ for isotropic material the right hand side is equal to zero

→ restore ordinary dispersion relation

→ eigenvalue equation that asks to solve the following characteristic equation

$$
\begin{bmatrix}
\frac{\omega^2}{c_0^2}\varepsilon_1 - k_2^2 - k_3^2 & k_1 k_2 & k_1 k_3 \\
k_2 k_1 & \frac{\omega^2}{c_0^2}\varepsilon_2 - k_1^2 - k_3^2 & k_2 k_3 \\
k_3 k_1 & k_3 k_2 & \frac{\omega^2}{c_0^2}\varepsilon_3 - k_1^2 - k_2^2
\end{bmatrix}
\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

→ determinant of this linear system is zero

→ from det[…]=0 we obtain dispersion relation $\omega = \omega(k)$ for a given ratio of k_i/k

one equation with four free parameters

fix three of these parameters and compute the fourth parameter

problem solved

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Approach 4: Derivation of the dispersion relation: sophisticated approach

wave equation in principal coordinate system and in component notation

$$\left(\frac{\omega^2}{c_0^2}\varepsilon_i - k^2\right)E_i = -k_i\sum_{j=1}^{3} k_j E_j$$

$$E_i = -\frac{k_i}{\left(\frac{\omega^2}{c_0^2}\varepsilon_i - k^2\right)}\sum_{j=1}^{3} k_j E_j$$

→ multiplication with k_i, summation over i, substitution between i and j on lhs

$$\sum_{j=1}^{3} k_j E_j = -\sum_{i=1}^{3}\frac{k_i^2}{\left(\frac{\omega^2}{c_0^2}\varepsilon_i - k^2\right)}\sum_{j=1}^{3} k_j E_j$$

→ divergence of the electric field is not vanishing: $\mathbf{div}\,\mathbf{E} = \sum_{j=1}^{3} k_j E_j \neq 0$

preliminary dispersion relation

$$\sum_{i=1}^{3}\frac{k_i^2}{\left(k^2 - \frac{\omega^2}{c_0^2}\varepsilon_i\right)} = 1$$

by rewriting the wave vector as

$$\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix} = k(\omega) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \frac{\omega}{c_0} n(\omega) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\sum_{i=1}^{3} \frac{k_i^2}{\left(k^2 - \frac{\omega^2}{c_0^2} \varepsilon_i(\omega)\right)} = 1 \rightarrow \sum_{i=1}^{3} \frac{u_i^2}{\left(1 - \frac{\varepsilon_i}{n^2(\omega)}\right)} = 1$$

$$\boxed{\sum_{i=1}^{3} \frac{u_i^2}{(n^2(\omega) - \varepsilon_i(\omega))} = \frac{1}{n^2(\omega)}}$$

→ final dispersion relation → can calculate $n(\omega, u_1, u_2)$

explicit expression

$$\boxed{\begin{aligned} u_1^2(n^2 - \varepsilon_2)(n^2 - \varepsilon_3)n^2 + u_2^2(n^2 - \varepsilon_1)(n^2 - \varepsilon_3)n^2 + u_3^2(n^2 - \varepsilon_1)(n^2 - \varepsilon_2)n^2 \\ = (n^2 - \varepsilon_1)(n^2 - \varepsilon_2)(n^2 - \varepsilon_3) \end{aligned}}$$

→ quadratic equation in n^2 (n^6 terms are vanishing)

→ two solutions n_a and n_b and with this also , $k_{a,b} = \frac{\omega}{c_0} n_{a,b}$

normal modes have a polarisation in the electric displacement for which they are perpendicular on each other

Example

$\rightarrow$ propagation direction along one of the principal axes ($u_3 = 1$)

$$(n^2 - \varepsilon_1)(n^2 - \varepsilon_2)n^2 = (n^2 - \varepsilon_1)(n^2 - \varepsilon_2)(n^2 - \varepsilon_3)$$

solutions: $n_a^2 = \varepsilon_1 \quad \text{and} \quad n_b^2 = \varepsilon_2$

computing the normal modes:

(starting point again wave equation)

$$\left(\frac{\omega^2}{c_0^2}\varepsilon_i - k^2\right)E_i = -k_i \sum_{j=1}^{3} k_j E_j \quad \rightarrow \quad \boxed{E_i = -\frac{k_i}{\left(\frac{\omega^2}{c_0^2}\varepsilon_i - k^2\right)} \sum_{j=1}^{3} k_j E_j}$$

sum is independent on *i*

$\rightarrow$ from $\sum_{j=1}^{3} k_j E_j = const$ we can write down the ratio of the amplitudes as

$$\boxed{E_1 : E_2 : E_3 = \frac{k_1}{\left(\frac{\omega^2}{c_0^2}\varepsilon_1 - k^2\right)} : \frac{k_2}{\left(\frac{\omega^2}{c_0^2}\varepsilon_2 - k^2\right)} : \frac{k_3}{\left(\frac{\omega^2}{c_0^2}\varepsilon_3 - k^2\right)}}$$

In combination with $D_i = \varepsilon_0 \varepsilon_i E_i$ we obtain

$$D_1 : D_2 : D_3 = \frac{\varepsilon_1 k_1}{\left(\frac{\omega^2}{c_0^2}\varepsilon_1 - k^2\right)} : \frac{\varepsilon_2 k_2}{\left(\frac{\omega^2}{c_0^2}\varepsilon_2 - k^2\right)} : \frac{\varepsilon_3 k_3}{\left(\frac{\omega^2}{c_0^2}\varepsilon_3 - k^2\right)}$$

→ field components are real valued

→ no phase difference between the different components

→ linear polarization for the normal modes.

→ modes $\mathbf{D}^{(a)}$ and $\mathbf{D}^{(b)}$ are orthogonal

Normal surfaces

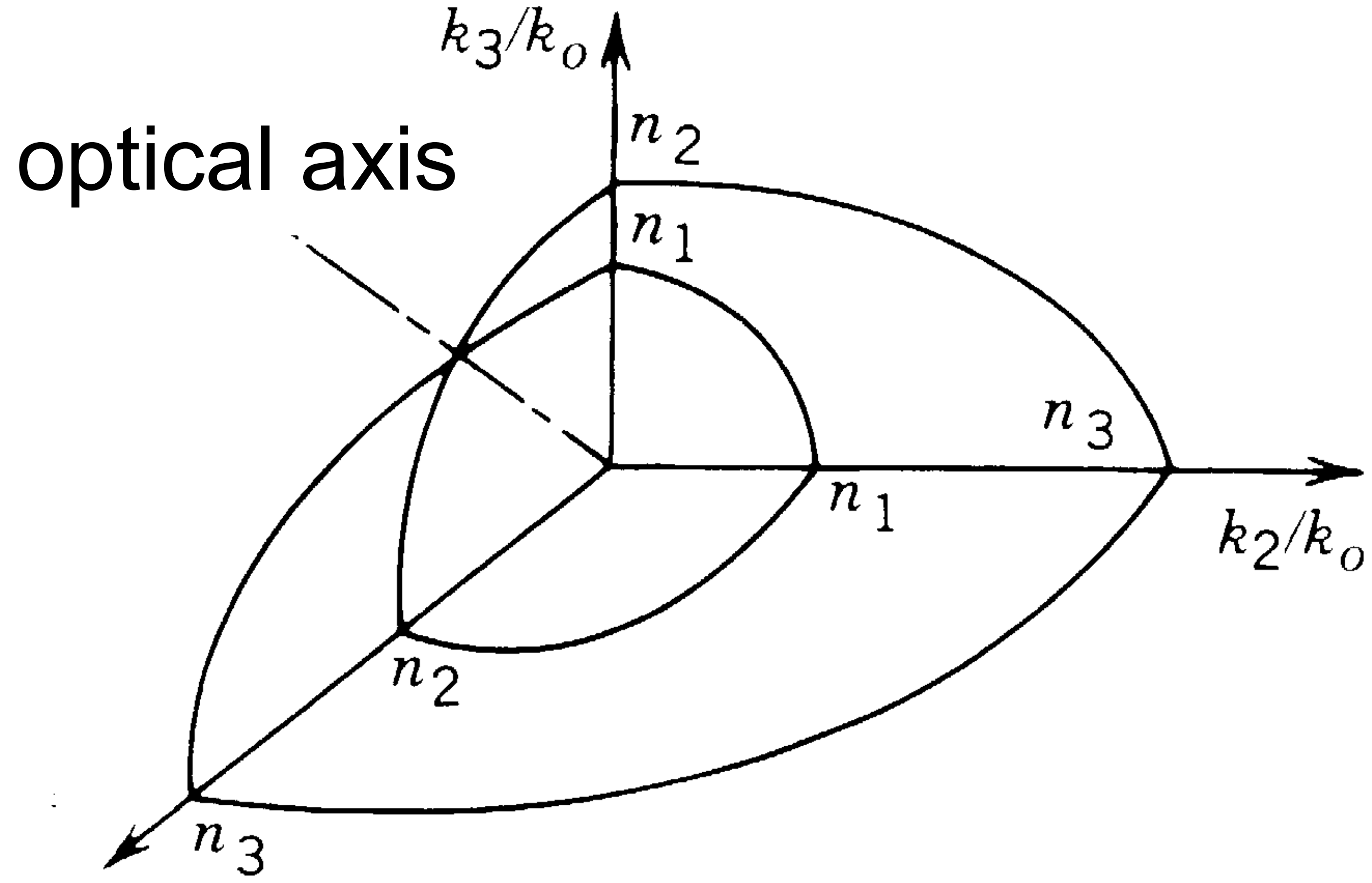
→ also called iso-frequency surface or iso-surface

→ plot the index of the two modes as surfaces depending on k_i

→ centro-symmetric two-layer surface

→ cross sections with principal axes are either circles or ellipses

biaxial crystal

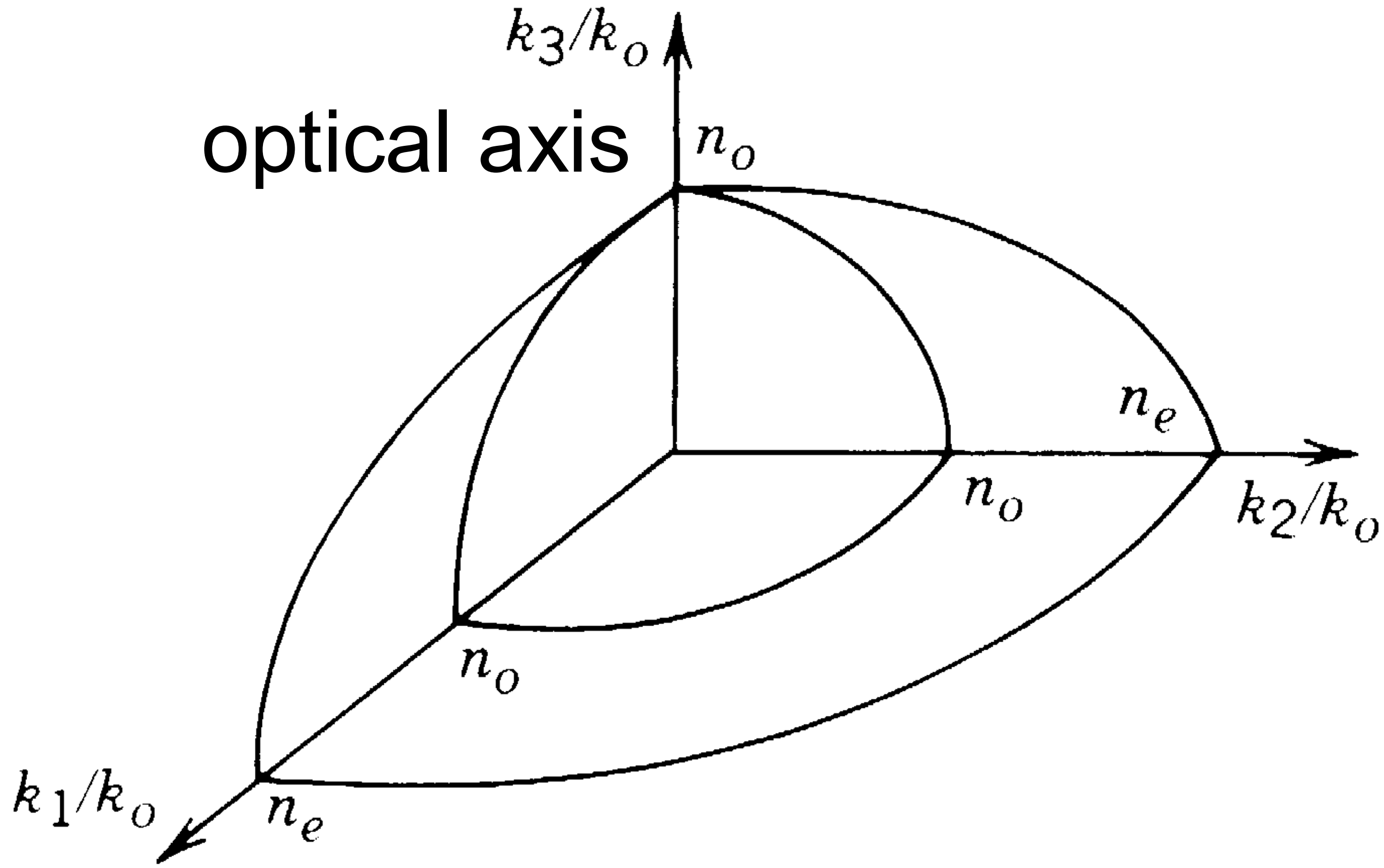


→ the two surfaces intersect in four different points

→ connecting lines between the two points are the two **optical axes**

→ optical axis defined as direction where the wave experiences no birefringence

uniaxial crystal



→ body of revolution made from an ellipse and a sphere

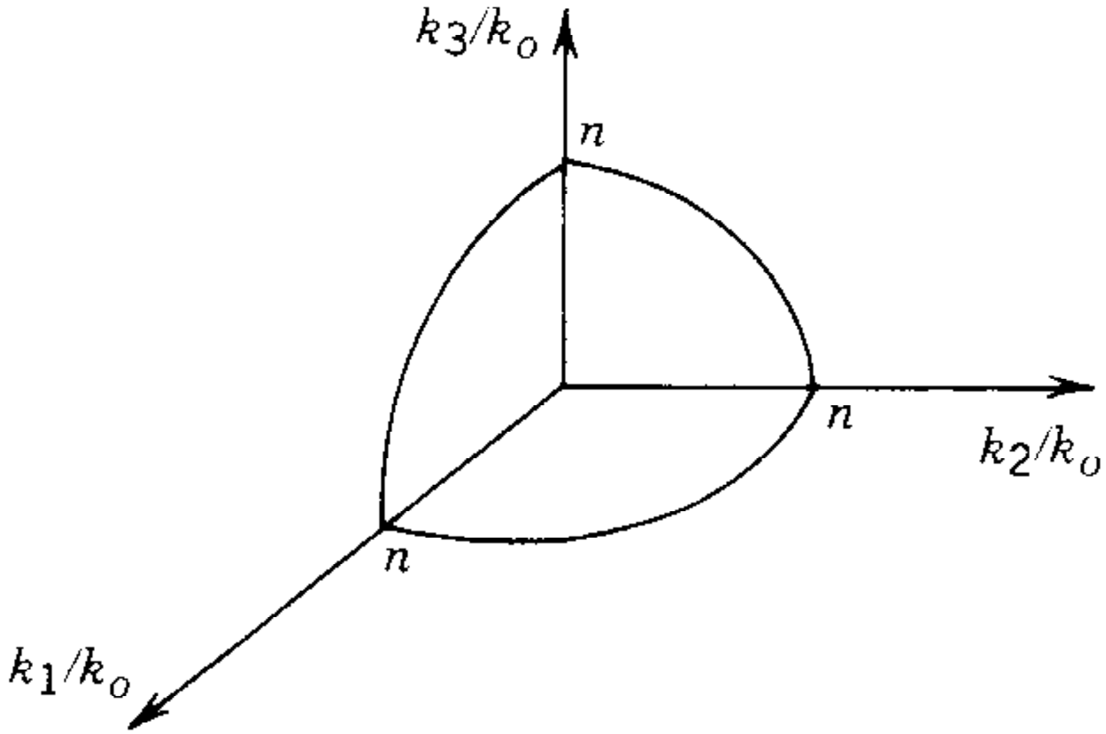
→ two intersection points at the poles

→ connecting line equally provides information on the optical axes (z)

→ $\varepsilon_1 = \varepsilon_2 = \varepsilon_{\mathrm{or}}$ and $\varepsilon_3 = \varepsilon_{\mathrm{e}}$

→ subscripts or and e stands for ordinary and extraordinary optical axes

cubic crystal



the structure is isotropic and the two interfaces are identical

how to use normalsurfaces

→ fix the directions u_1 and u_2

→ identify the intersections with the surfaces

→ distance between from coordinate to intersections provides refractive indices of the normal modes

→ only if considered along optical axes, the two indices are identical $n_a = n_b$

summary of the two geometrical interpretation

(a) Index ellipsoid

-Direction fixed → identifying index ellipse → semi-axes provide n_a and n_b, being the indices, which are experienced by the normal modes

(b) Normal surfaces

-Direction fixed → cross section to the normal surfaces → distance to the center provide n_a and n_b, optical axis is the connecting line between the center and the cross section of the two branches

essence

→ two linearly polarised monochromatic waves as normal modes

→ have different phase velocities, given by $\frac{c_0}{n_{a,b}}$

→ two perpendicular polarization direction

can be extracted once material and propagation direction are fixed

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