Theoretical Optics

Coherence Optics

Statistical properties of light

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Definitions

$u_{\mathrm{R}}(\mathbf{r}, t)$ physical component of a non-monochromatic em-field at a certain position r and time t

real valued

$u(\mathbf{r}, t)$ associated complex analytical signal

$$u(\mathbf{r},t) = \int_0^\infty U(\mathbf{r},\omega)e^{-i\omega t}d\omega \qquad u_R(\mathbf{r},t) = \Re\left[\int_0^\infty U(\mathbf{r},\omega)e^{-i\omega t}d\omega\right]$$

Intensity

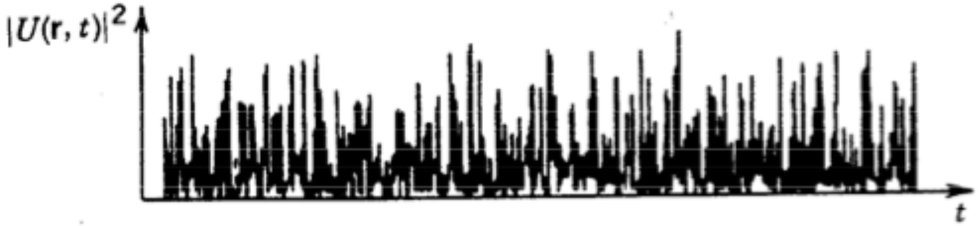
$$I(\mathbf{r},t) = \langle |u(\mathbf{r},t)|^2 \rangle_e$$

randomly fluctuating instantaneous intensity

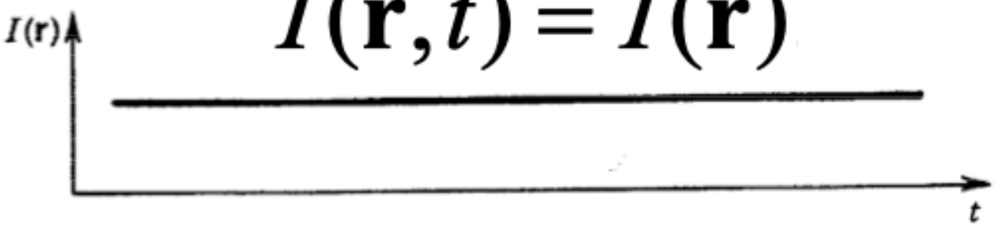
ensemble average

many measurements on nominally identically prepared systems

example 1: stationary statistical light source
(light valve driven by constant current)



$$I(\mathbf{r},t)=I(\mathbf{r})$$



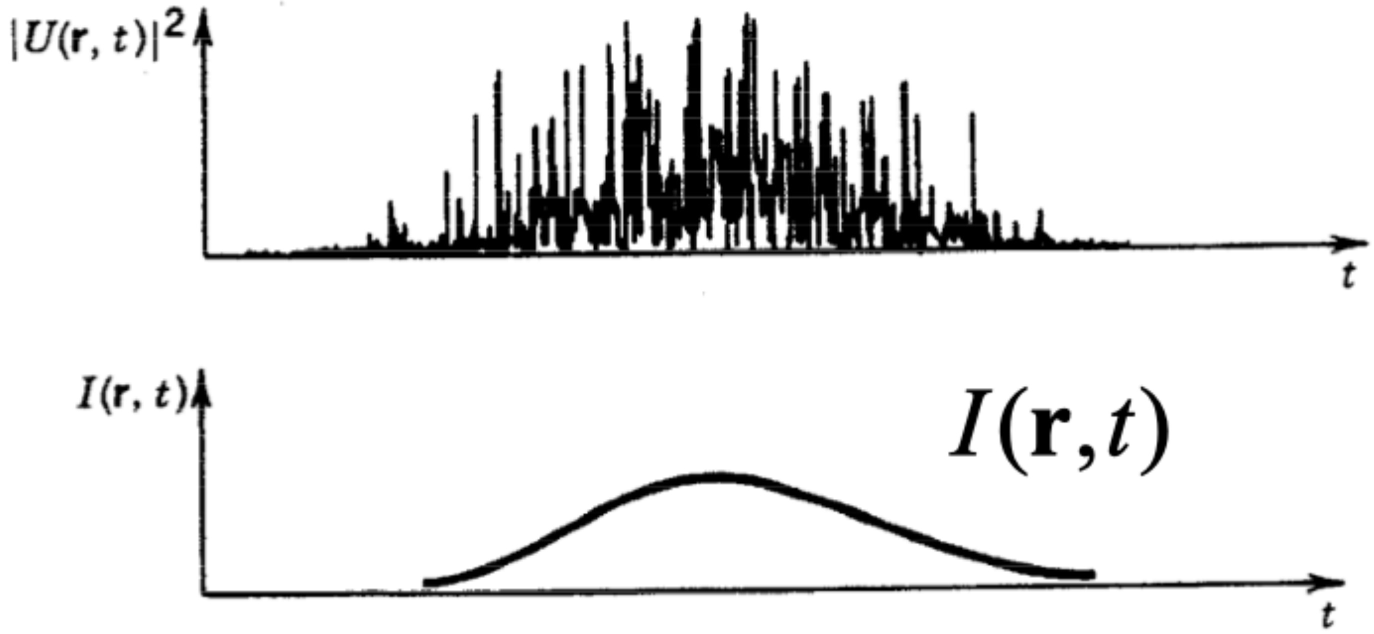
!!ergodic hypothesis applies!!

averaged over time = averaged over ensembles

$$\langle\cdots\rangle_e = \langle\cdots\rangle_t = \langle\cdots\rangle$$

$$\boxed{I(\mathbf{r}) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|u(\mathbf{r},t)|^2 dt}$$

example 2: non-stationary statistical light source
(light valve driven by time dependent current)



$I(\mathbf{r},t)$

Temporal coherence and spectrum

consider stationary statistical process: $\longrightarrow$ $I(\mathbf{r})$

consider fix point in space: $\longrightarrow$ $I(\mathbf{r}) = I$

measure for correlation of the field at $u(t)$ and $u(t+\tau)$

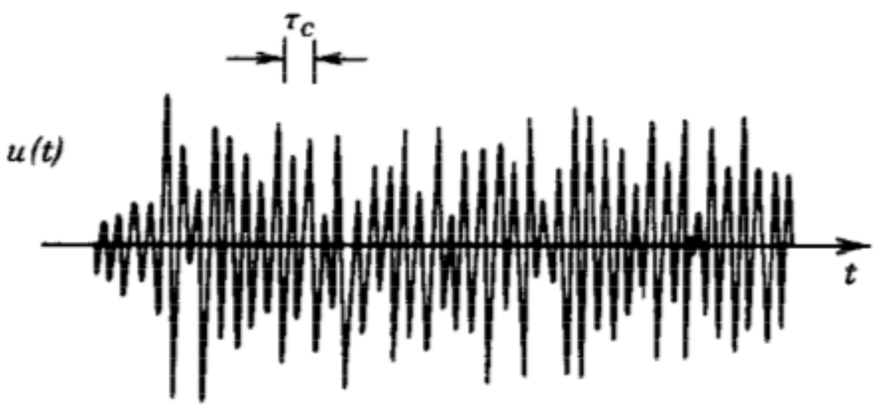
auto-correlation function

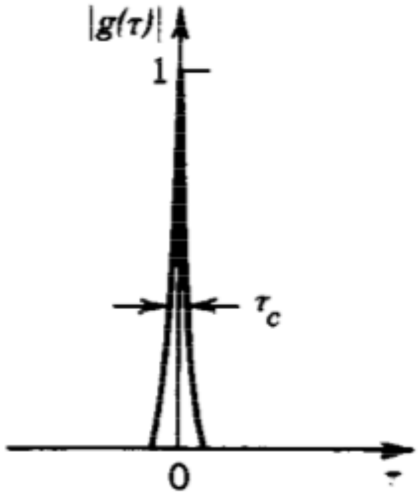
$$G(\tau) = \langle u^*(t)u(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} u^*(t)u(t+\tau)dt$$

temporal coherence (self-coherence) function

temporal width: time scale of change of envelope

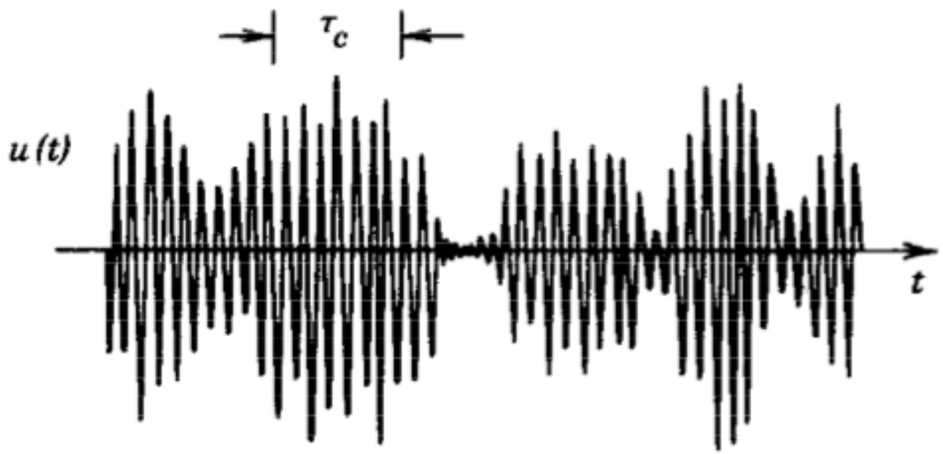
coherence time τ_C

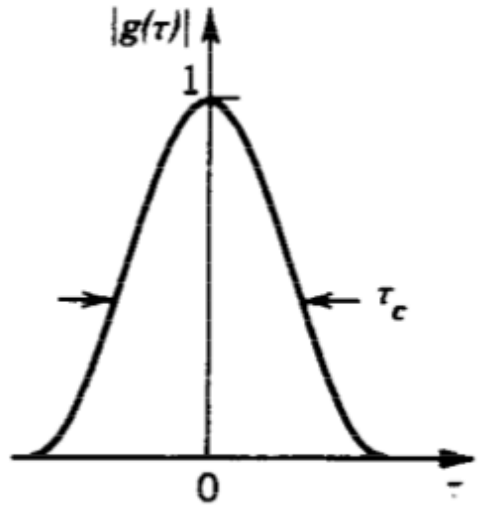




beyond this time, coherence function uncorrelated $G(\tau) = 0$

it holds: $\boxed{G(0) = I}$





function contains 2 information (intensity and coherence)

normalisation ⟶ complex degree of temporal coherence

$$\boxed{g(\tau) = \frac{G(\tau)}{G(0)} = \frac{\langle u^*(t)u(t+\tau)\rangle}{\langle u^*(t)u(t)\rangle}} \quad 0 \leq g(\tau) \leq 1$$

example 1: coherent light $u(t) = Ae^{-i\omega_0 t}$

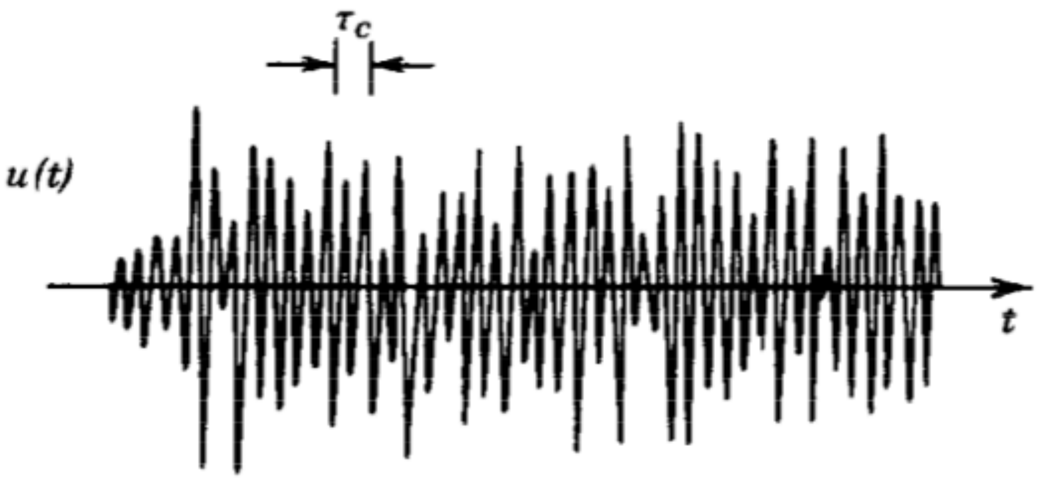
$$g(\tau) = \frac{\frac{1}{2T}\int_{-T}^{T}|A|^2 e^{-i\omega_0\tau}dt}{\frac{1}{2T}\int_{-T}^{T}|A|^2 dt} = e^{-i\omega_0\tau} \rightarrow \boxed{|g(\tau)| = 1}$$

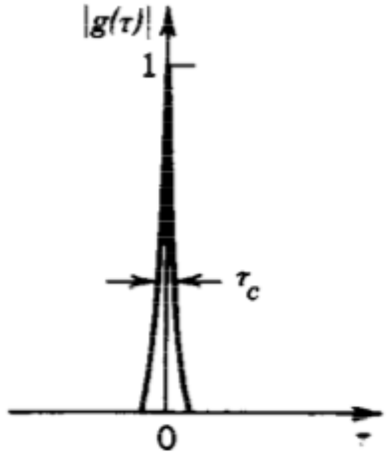
example 2: general case

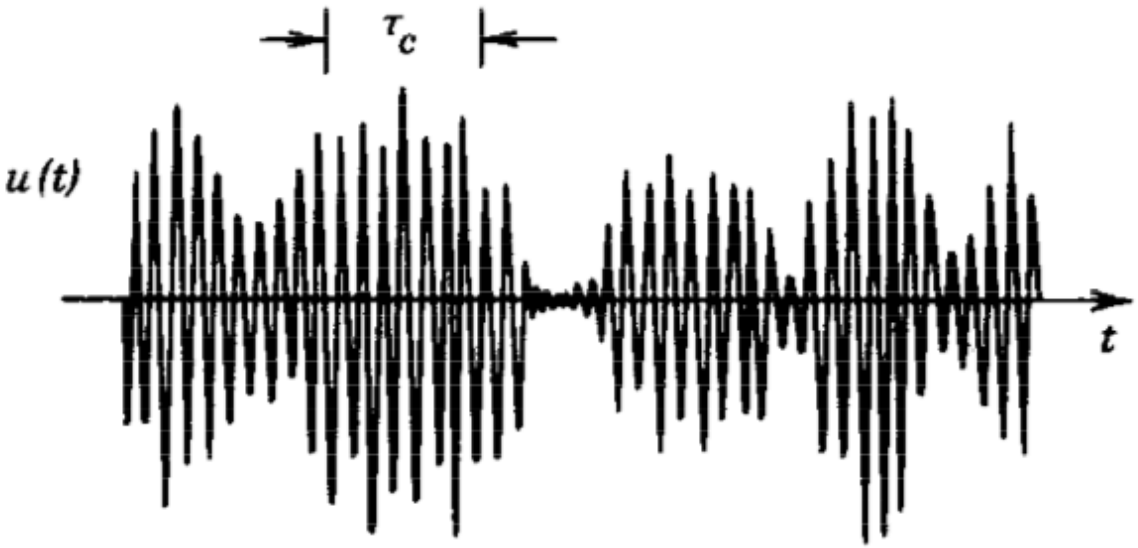
$|g(\tau)|$ is a monotonously decaying function

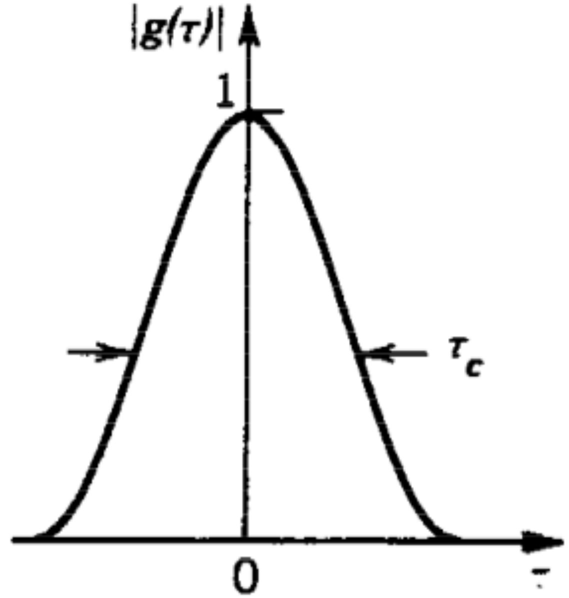
coherence time: $|g(\tau_c)| = 1/e$

$$\boxed{\tau_c = \int_{-\infty}^{\infty}|g(\tau)|^2 d\tau}$$









coherent light: $|g(\tau)| = 1 \quad \curvearrowright \quad \tau_c \to \infty$

general example: $g(\tau) = e^{-\frac{|\tau|}{A}}$

$$\longrightarrow \tau_c = 2\int_0^{\infty} e^{-2\frac{|\tau|}{A}} d\tau = A(1-0) = A$$

equivalent coherence length: $l_c = c\tau_c$

all length scales of optical system < than coherence length

$\longrightarrow$ system coherent

Theorem of Wiener-Khinchin

dependency between spectrum and temporal coherence

$$G(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{-i\omega\tau} d\omega$$

signal: $S(\omega) = |U(\omega)|^2$

Fourier transformation of the temporal coherence function corresponds to signal spectrum

<u>proof</u>: $G(\tau) = \left\langle u^*(t) u(t+\tau) \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u^*(t) u(t+\tau) dt$

finite FT

$$U_T(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u_T(t) \exp(i\omega t) dt \quad u_T(t) = \begin{cases} u(t) & \text{for } |t| \le T \\ 0 & \text{otherwise} \end{cases}$$

$$u_{T}(t) = \int_{-\infty}^{\infty} U_{T}(\omega) \exp(-\mathbf{i}\omega t) d\omega$$

$$G(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-T}^{T} U_T^{\star}(\omega') e^{i\omega' t} U_T(\omega'') e^{-i\omega''(t+\tau)} d\omega' d\omega'' dt$$

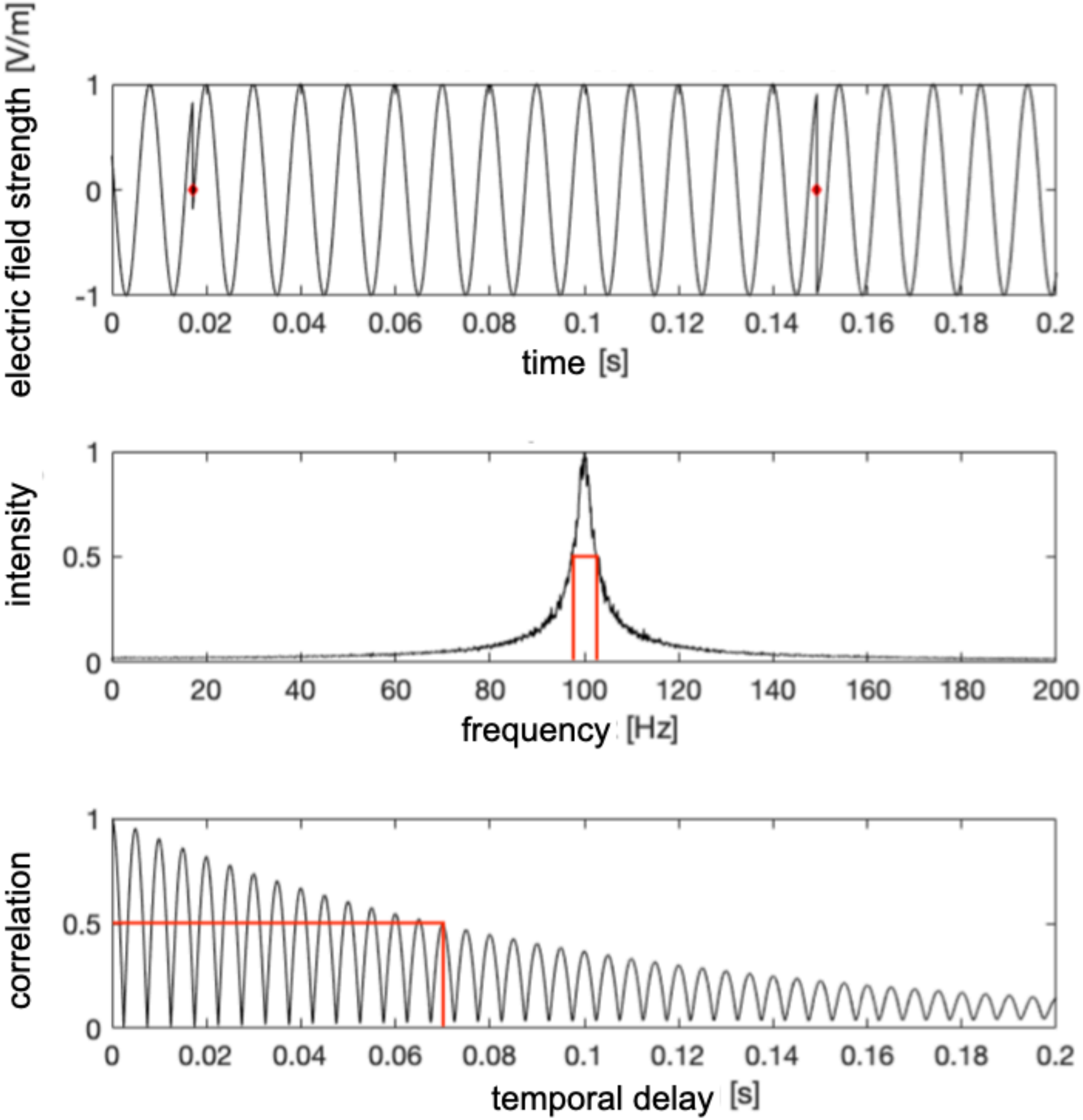
$$= \lim_{T\to\infty} \frac{1}{2T} 2\pi \int_{-\infty}^{\infty} |U_T(\omega')|^2 e^{-i\omega'\tau} d\omega'$$

$$\curvearrowright S(\omega) = 2\pi \lim_{T\to\infty} \frac{1}{2T} \left|U_{T}(\omega)\right|^2$$

$S(\omega)d\omega$ $\rightarrow$ spectral intensity in the interval between ω and $\omega + d\omega$

$$G(0) = I \quad \longrightarrow \quad I = \int_0^{\infty} S(\omega) d\omega$$

coherence of light source is given by spectral intensity



electric field strength [V/m]
1
0
-1
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
time [s]
intensity
1
0.5
0
0
20
40
60
80
100
120
140
160
180
200
frequency [Hz]
correlation
1
0.5
0
0
0.02
0.04
0.06
0.08
0.1
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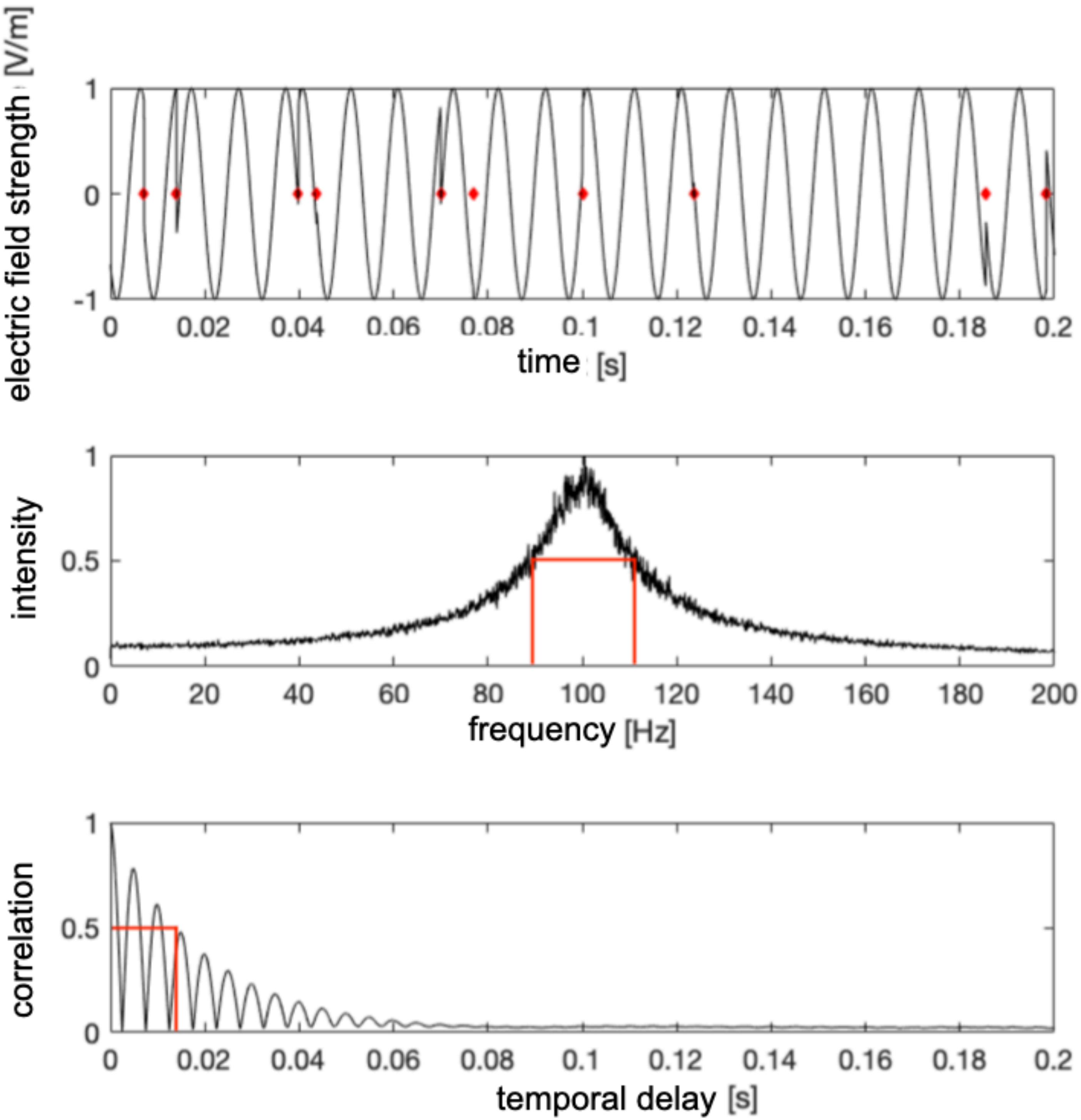
temporal delay [s]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0.12
0.14
0.16
0.18
0.2
temporal delay [s]

product between coherence time and bandwidth is constant

$$\tau_c \Delta\omega = \text{const.}$$

spectral width:

$$\Delta\omega = \frac{\left[\int_0^\infty S(\omega)d\omega\right]^2}{\int_0^\infty S^2(\omega)d\omega}$$

spectral filter can improve coherence

loss of intensity

TABLE 10.1-2 Spectral Widths of a Number of Light Sources Together with Their Coherence Times and Coherence Lengths in Free Space

Source	$\Delta\nu_c$ (Hz)	$\tau_c = 1/\Delta\nu_c$	$l_c = c\tau_c$
Filtered sunlight (λ_o = 0.4–0.8 μm)	3.75×10^{14}	2.67 fs	800 nm
Light-emitting diode (λ_o = 1 μm, $\Delta\lambda_o$ = 50 nm)	1.5×10^{13}	67 fs	20 μm
Low-pressure sodium lamp	5×10^{11}	2 ps	600 μm
Multimode He–Ne laser (λ_o = 633 nm)	1.5×10^{9}	0.67 ns	20 cm
Single-mode He–Ne laser (λ_o = 633 nm)	1×10^{6}	1 μs	300 m

Mutual coherence function

(spatial and temporal coherence)

cross-correlation at 2 spatial and 2 temporal positions

$u(\mathbf{r}_1, t_1)$ and $u(\mathbf{r}_2, t_2)$ with $\tau = t_2 - t_1$

$$G(\mathbf{r}_1, \mathbf{r}_2, \tau) = \langle u^*(\mathbf{r}_1, t) u(\mathbf{r}_2, t + \tau) \rangle$$

complex degree of mutual coherence

$$\boxed{g(\mathbf{r}_1, \mathbf{r}_2, \tau) = \frac{G(\mathbf{r}_1, \mathbf{r}_2, \tau)}{\sqrt{I(\mathbf{r}_1) I(\mathbf{r}_2)}}}$$

$$0 \le |g(\mathbf{r}_1, \mathbf{r}_2, \tau)| \le 1$$

example: plane monochromatic wave

$$|g(\mathbf{r}_1, \mathbf{r}_2, \tau)| = \left|e^{i\mathbf{k}(\mathbf{r}_1 - \mathbf{r}_2) - i\omega\tau}\right| = 1$$

special: temporal coherence $\mathbf{r}_1 = \mathbf{r}_2 = \mathbf{r}$

spatial coherence $\tau = 0$

Spatial coherence function

spatial case of mutual coherence function while assuming $\tau = 0$

$$G(\mathbf{r}_1, \mathbf{r}_2, 0) = \langle u^*(\mathbf{r}_1, t) u(\mathbf{r}_2, t) \rangle = G(\mathbf{r}_1, \mathbf{r}_2)$$

⟶ normalised mutual intensity $$g(\mathbf{r}_1, \mathbf{r}_2) = \frac{G(\mathbf{r}_1, \mathbf{r}_2)}{\sqrt{I(\mathbf{r}_1) I(\mathbf{r}_2)}}$$

Van Cittert–Zernike theorem: equivalent to Wiener-Khinchin theorem in spatial domain

relation between spatial degree of coherence $|g(\mathbf{r}_1,\mathbf{r}_2)|$
and intensity distribution of an aperture $I(x',y')$

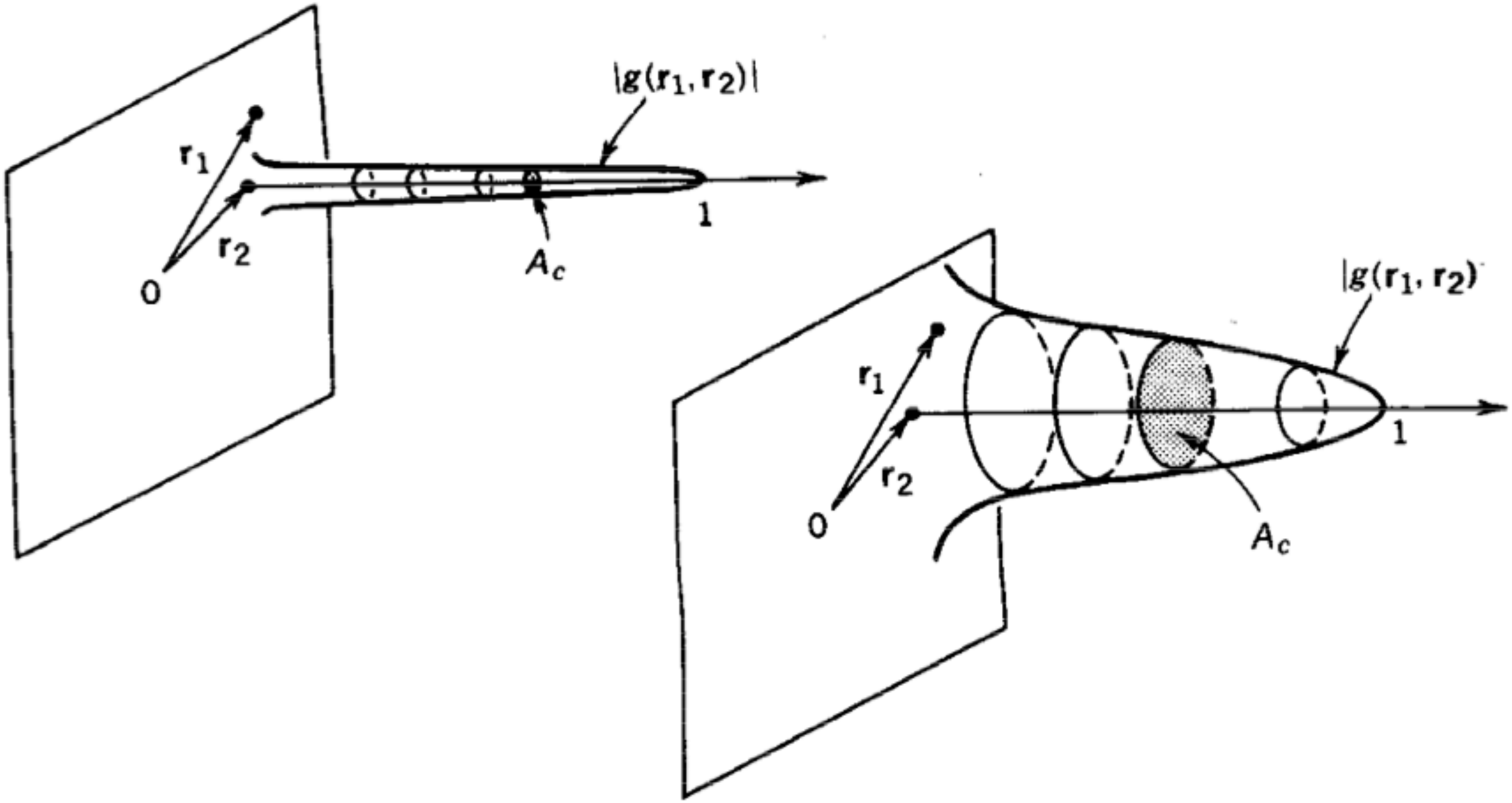
$$|g(\mathbf{r}_1,\mathbf{r}_2)| = \left| e^{ik\frac{(x_1^2+y_1^2)-(x_2^2+y_2^2)}{2z}} \cdot \frac{\iint_\sigma I(x',y') \cdot e^{-i2\pi(px'+qy')} \mathrm{d}x'\mathrm{d}y'}{\iint_\sigma I(x',y')\,\mathrm{d}x'\mathrm{d}y'} \right|$$

$$p = \frac{k}{2\pi}\frac{x_1-x_2}{z} \qquad q = \frac{k}{2\pi}\frac{y_1-y_2}{z}$$

mutual intensity= prefactor • $\frac{\text{Fourier transform of intensity in aperture}}{\text{normalisation}}$

circular aperture:

$$|g(\mathbf{r}_1,\mathbf{r}_2)| \propto \left| 2J_1\left(\frac{2\pi\cdot\rho\cdot|\mathbf{r}_1-\mathbf{r}_2|}{\lambda_0\cdot z}\right) \Big/ \left(\frac{2\pi\cdot\rho\cdot|\mathbf{r}_1-\mathbf{r}_2|}{\lambda_0\cdot z}\right)\right|$$



mean spatial extent within which the field is coherent $J_1\left(\frac{2\pi\cdot\rho\cdot|\mathbf{r_1}-\mathbf{r}_2|}{\lambda_0\cdot z}\right) = 0 \longrightarrow |\mathbf{r_1}-\mathbf{r}_2| = 1,22\cdot\frac{\lambda_0\cdot z}{2\cdot\rho}$

mean size of the coherent area $\bar{x} = 1,22\cdot\frac{\lambda_0\cdot z}{2\cdot\rho} \longrightarrow$

$$\boxed{\bar{x} = 1,22\cdot\frac{\lambda_0}{2\cdot\tan\left(\frac{\alpha}{2}\right)}}$$

angular extent of light source

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