Coherence Optics
Propagation of partially coherent light

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Evolution of mutual coherence function

show that mutual coherence function obeys 2 PDE

$\longrightarrow$ propagation of mutual coherence from one plane to another just as the electromagnetic field propagates

consider two fields $u_1(\mathbf{r},t)$ and $u_2(\mathbf{r},t)$

$$\triangle u_1(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2 u_1(\mathbf{r},t)}{\partial t^2} \qquad \triangle u_2(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2 u_2(\mathbf{r},t)}{\partial t^2}$$

mutual coherence function:

$$G_{12}(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2) = \langle u_1(\mathbf{r}_1,t-t_1)u_2^*(\mathbf{r}_2,t-t_2)\rangle$$

$$\begin{aligned}\triangle_1 G_{12}(\mathbf{r}_1,\mathbf{r}_2,t_1,t_2) &= \triangle_1\langle u_1(\mathbf{r}_1,t-t_1)u_2^*(\mathbf{r}_2,t-t_2)\rangle \\ &= \langle\left[\triangle_1 u_1(\mathbf{r}_1,t-t_1)\right] u_2^*(\mathbf{r}_2,t-t_2)\rangle\end{aligned}$$

by using the wave equation we obtain

$$\triangle_1 G_{12}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \frac{1}{c^2}\left\langle \frac{\partial^2 u_1(\mathbf{r}_1, t - t_1)}{\partial (t - t_1)^2} u_2^*(\mathbf{r}_2, t - t_2) \right\rangle$$

$$= \frac{1}{c^2}\left\langle \frac{\partial^2 u_1(\mathbf{r}_1, t - t_1)}{\partial t_1^2} u_2^*(\mathbf{r}_2, t - t_2) \right\rangle$$

$$\boxed{\triangle_1 G_{12}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \frac{1}{c^2}\frac{\partial^2}{\partial t_1^2} G_{12}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)}$$

in analogy it also holds that

$$\boxed{\triangle_2 G_{12}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \frac{1}{c^2}\frac{\partial^2}{\partial t_2^2} G_{12}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)}$$

two PDEs the mutual coherence function obeys

Specification for stationary fields

$$\tau = t_2 - t_1 \quad \longrightarrow \quad \frac{\partial^2}{\partial t_1^2} = \frac{\partial^2}{\partial t_2^2} = \frac{\partial^2}{\partial \tau^2}$$

$$\boxed{\triangle_2 G_{12}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \triangle_1 G_{12}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2) = \frac{1}{c^2}\frac{\partial^2}{\partial \tau^2} G_{12}(\mathbf{r}_1, \mathbf{r}_2, t_1, t_2)}$$

expresses space-time-evolution of coherence function

Interference of partially coherent light

consider interference at one point

two randomly fluctuating fields $u_1(\mathbf{r}, t)$ and $u_2(\mathbf{r}, t)$

intensity corresponds to $I_1 = \langle |u_1(\mathbf{r}, t)|^2 \rangle$ and $I_2 = \langle |u_2(\mathbf{r}, t)|^2 \rangle$

characterised by cross correlation function $G_{12} = \langle u_1^* u_2 \rangle$

mutual complex degree of coherence: $g_{12} = \dfrac{\langle u_1^* u_2 \rangle}{\sqrt{I_1 I_2}}$

interference between both fields

$$I(\mathbf{r}) = \left\langle \left| u_1 + u_2 \right|^2 \right\rangle = \left\langle \left| u_1 \right|^2 \right\rangle + \left\langle \left| u_2 \right|^2 \right\rangle + \left\langle u_1^* u_2 \right\rangle + \left\langle u_1 u_2^* \right\rangle$$

$$= I_1 + I_2 + G_{12} + G_{12}^* = I_1 + I_2 + 2\Re\left(G_{12}\right)$$

$$= I_1 + I_2 + 2\sqrt{I_1 I_2}\,\Re\left(g_{12}\right)$$

$$= I_1 + I_2 + 2\sqrt{I_1 I_2}\,\left|g_{12}\right| \cos\varphi, \qquad \varphi = \arg\left(g_{12}\right)$$

consequences: $|g_{12}| = 1 \longrightarrow$ perfect interference

$|g_{12}| = 0 \rightarrow I = I_1 + I_2$ no interference

$0 \leq |g_{12}| \leq 1 \longrightarrow$ interference with some visibility/contrast

$$\nu = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = 2\,\frac{\sqrt{I_1 I_2}}{I_1 + I_2}\,|g_{12}|$$

special $\quad I_1 = I_2 \longrightarrow \boxed{\nu = |g_{12}|}$

basis for a measurement procedure

Interference and temporal coherence

signal at same location but different time (stationary field)

$$u_1 = u(t) \quad \rightarrow \quad I_0 = \langle |u(t)|^2 \rangle$$

$$u_2 = u(t+\tau) \quad \rightarrow \quad I_0 = \langle |u(t+\tau)|^2 \rangle$$

corresponds to a time delayed wave

degree of coherence: $g_{12} = \frac{\langle u_1^* u_2 \rangle}{I_0} = \frac{\langle u^*(t) u(t+\tau) \rangle}{I_0} = g(\tau)$

interference pattern: $I(\tau) = 2I_0\{1 + \Re(g(\tau))\}$

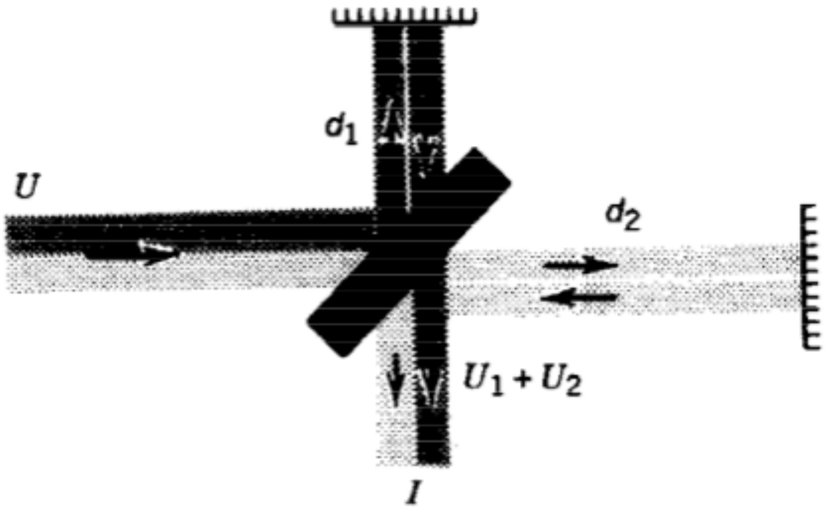
$$= 2I_0\{1 + |g(\tau)| \cos \phi(\tau)\}$$

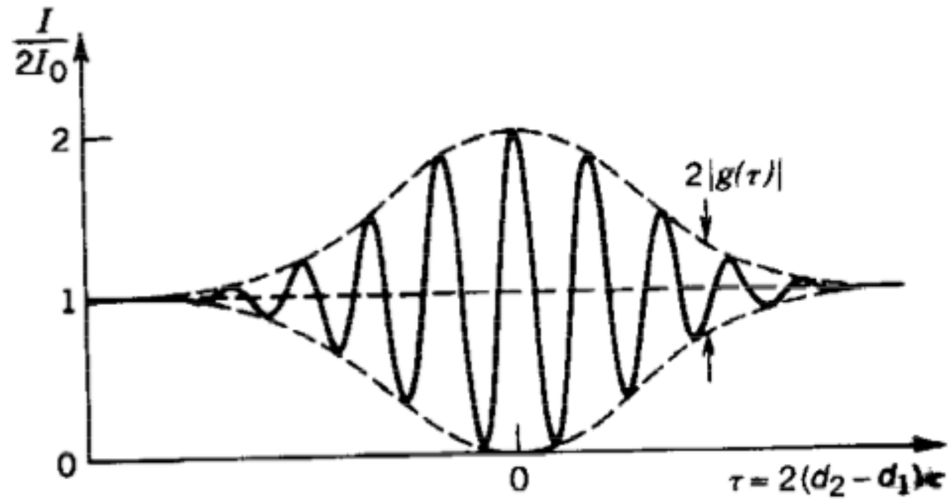
defines interference fringes

visibility: $\nu = |g(\tau)|$

perfect contrast: $(\nu = 1)$ for $\tau = 0$ vanishing contrast: $(\nu = 0)$ for $\tau > \tau_c$

measurement in Michelson or Mach-Zehnder interferometer





path difference in arms dictates time delay: $\tau = \frac{2(d_2 - d_1)}{c}$

consider quasi-monochromatic light $(\omega_0 \gg \Delta\omega)$

$$u(t) = a(t)e^{-i\omega_0 t}$$

$$g(\tau) = g_a(\tau)e^{-i\omega_0\tau} = |g_a(\tau)|e^{-i(\omega_0\tau - \varphi_a)}$$

$$g_a(\tau) = \frac{\langle a^*(t)a(t+\tau)\rangle}{|a(t)|^2}$$

$$I(\tau) = 2I_0\{1 + |g_a(\tau)|\cos(\omega_0\tau - \phi_a(\tau))\}$$

contrast

oscillation period

absolute position of fringes

by measuring intensity, we can measure the absolute value of mutual temporal coherence

application: Fourier-Transform spectroscopy

spectrum is collected based on measurements
of the coherence of a radiative source

remember
from Wiener-Kintchin: $G(\tau) = g(\tau) I_0 = \int_0^\infty S(\omega) e^{-i\omega\tau} d\omega$

$$I(\tau) = 2I_0 + 2I_0 \Re g(\tau)$$

$$= 2\int_0^\infty S(\omega) d\omega + 2\Re \int_0^\infty S(\omega) \exp(-\mathbf{i}\omega\tau) d\omega$$

$$= 2\int_0^\infty S(\omega) d\omega + 2\int_0^\infty S(\omega) \cos(\omega\tau) d\omega$$

$$= 2\int_0^\infty S(\omega) [1 + \cos(\omega\tau)] d\omega$$

measure $I(\tau) \rightarrow$ inverse Fourier cosine transformation $\rightarrow S(\omega)$

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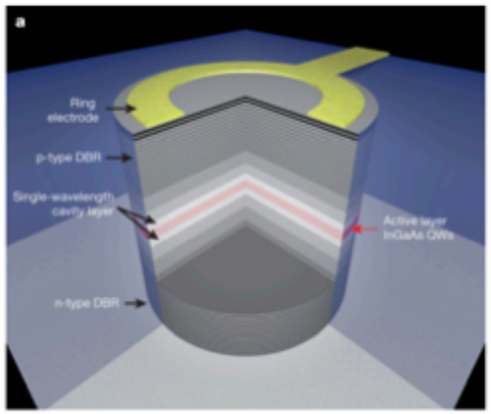
General idea

definition of a principal propagation direction

e.g. 1 D photonic crystals for periodic repetition of unit cell

- Bragg mirrors
- chirped mirrors for dispersion compensation
- interferometers



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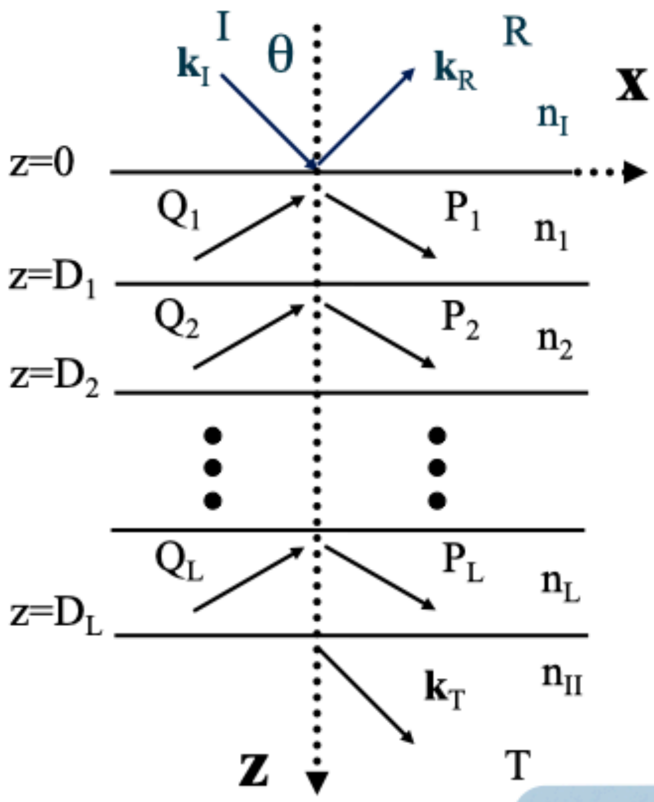
Reflection / Transmission at a stack of layers

assumption: slab consist of an arbitrary number of layers (TE polarisation)



fields in homogenous space have to be a solution to the wave equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\epsilon_f(x)\right]\mathbf{E}(x, z, \omega) = 0$$

invariant in x-direction justifies the use of plane waves in this direction in each region

ansatz:
$$\mathbf{E}(x, z) = \mathbf{E}(z)e^{ik_x x}$$
$$\mathbf{H}(x, z) = \mathbf{H}(z)e^{ik_x x}$$

$$\left[\frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2}\epsilon_f(\omega) - k_x^2\right]\mathbf{E}(z) = 0$$
$$\mathbf{H}(x, z) = -\frac{i}{\omega\mu_0}\mathrm{rot}\mathbf{E}(x, z)$$

Separating the problem into 2 polarisations

→ continuity of the tangential electric and magnetic field

tangential fields (TE): $E_y = E \quad H_x$

$$\left[\frac{\partial^2}{\partial z^2} + k_{fz}^2(k_x,\omega)\right] E(z) = 0 \qquad k_{fz}^2(k_x,\omega) = \frac{\omega^2}{c^2}\epsilon_f(\omega) - k_x^2$$

$$H_x(z) = -\frac{i}{\omega\mu_0}\frac{\partial}{\partial z}E(z) \qquad i\omega\mu_0 H_x(z) = \frac{\partial}{\partial z}E(z)$$

tangential fields (TM): $H_y = H \quad E_x$

$$\left[\frac{\partial^2}{\partial z^2} + k_{fz}^2(k_x,\omega)\right] H(z) = 0$$

$$E_x(z) = \frac{i}{\omega\epsilon_0\epsilon_f}\frac{\partial}{\partial z}H(z) \qquad -i\omega\epsilon_0 E_x(z) = \frac{1}{\epsilon_f}\frac{\partial}{\partial z}H(z)$$

Polarisation independent formulation

- calculating the fields and their normal derivatives at $z = d$
- need to know the respective values at $z = 0$
- equations are the same ⟹ simultaneous treatment

$$E, H \longrightarrow F$$

$$i\omega\mu_0 H_x, -i\omega\epsilon_0 E_x \longrightarrow G$$

$$\left[\frac{\partial^2}{\partial z^2} + k_{fz}^2(k_x, \omega)\right] F(z) = 0$$

$$G(z) = \alpha_f \frac{\partial}{\partial z} F(z)$$

$$\alpha_{f\mathrm{TE}} = 1$$

$$\alpha_{f\mathrm{TM}} = \frac{1}{\epsilon_f}$$

Solving the initial value problem

$$F(z) = C_1 e^{ik_{fz}z} + C_2 e^{-ik_{fz}z}$$

$$G(z) = \alpha_f \frac{\partial}{\partial z} F(z) = i\alpha_f k_{fz} \left[C_1 e^{ik_{fz}z} - C_2 e^{-ik_{fz}z} \right]$$

need to know constants C_1 and C_2

$F(0)$ and $G(0)$ are known $\Rightarrow$ C_1 and C_2

$$F(0) = C_1 + C_2 \qquad G(0) = i\alpha_f k_{fz} \left[C_1 - C_2 \right]$$

$$\Rightarrow \quad C_1 = \frac{1}{2}\left[F(0) - \frac{i}{\alpha_f k_{fz}} G(0) \right] \quad C_2 = \frac{1}{2}\left[F(0) + \frac{i}{\alpha_f k_{fz}} G(0) \right]$$

A single transfer matrix

$$F(z) = \cos\left(k_{fz} z\right) F(0) + \frac{1}{\alpha_f k_{fz}} \sin\left(k_{fz} z\right) G(0)$$

$$G(z) = -\alpha_f k_{fz} \sin\left(k_{fz} z\right) F(0) + \cos\left(k_{fz} z\right) G(0)$$

writing the equations in matrix form

$$\begin{Bmatrix} F(z) \\ G(z) \end{Bmatrix} = \hat{\mathbf{m}} \begin{Bmatrix} F(0) \\ G(0) \end{Bmatrix}$$

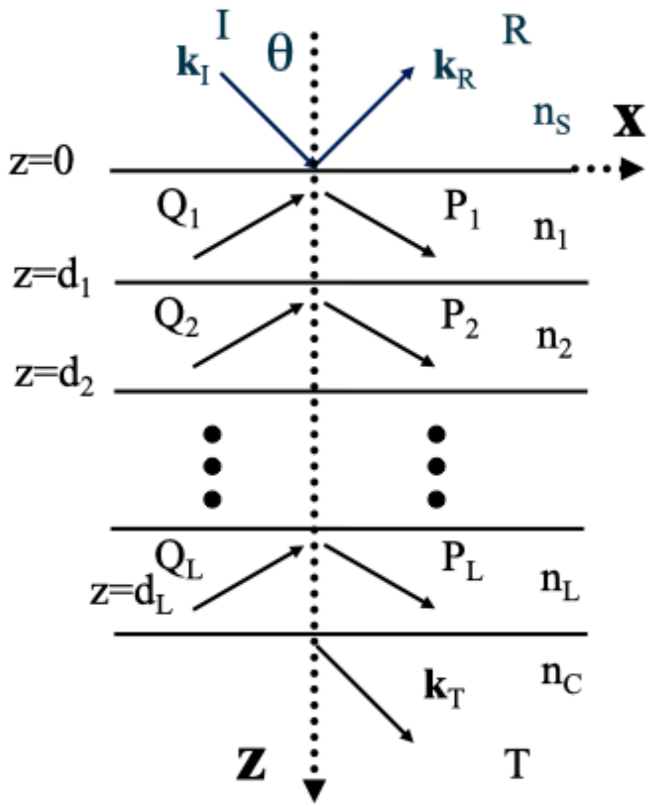
$$\hat{\mathbf{m}}(z) = \begin{Bmatrix} \cos\left(k_{fz} z\right) & \frac{1}{k_{fz}\alpha_f} \sin\left(k_{fz} z\right) \\ -k_{fz}\alpha_f \sin\left(k_{fz} z\right) & \cos\left(k_{fz} z\right) \end{Bmatrix}$$

single transfer matrix

Transfer matrix of a stack

N-layers: $$\begin{pmatrix} F \\ G \end{pmatrix}_{d_1+d_1+..+d_N=D} = \prod_{i=1}^{N} \hat{\mathbf{m}}_{\mathbf{i}}\left(d_i\right) \begin{pmatrix} F \\ G \end{pmatrix}_0 = \hat{\mathbf{M}} \begin{pmatrix} F \\ G \end{pmatrix}_0$$

Incident and transmitted field



$$\mathbf{k}_I = \begin{pmatrix} k_x \\ 0 \\ k_{sz} \end{pmatrix} \mathbf{k}_R = \begin{pmatrix} k_x \\ 0 \\ -k_{sz} \end{pmatrix} \mathbf{k}_T = \begin{pmatrix} k_x \\ 0 \\ k_{cz} \end{pmatrix}$$

$$\boxed{\begin{aligned} k_{cz} &= \sqrt{\frac{\omega^2}{c^2}\epsilon_c(\omega) - k_x^2} \\ k_{sz} &= \sqrt{\frac{\omega^2}{c^2}\epsilon_s(\omega) - k_x^2} \end{aligned}}$$

Coupling an incident field

ansatz for the field in the substrate:

$$F_s(x,z) = e^{ik_x x}\left[F_I e^{ik_{sz}z} + F_R e^{-ik_{sz}z}\right]$$

$$G_s(x,z) = i\alpha_s k_{sz} e^{ik_x x}\left[F_I e^{ik_{sz}z} - F_R e^{-ik_{sz}z}\right]$$

ansatz for the field in the stratified media:
(known from matrix method)

$$F_f(x,z) = e^{ik_x x}F(z) \qquad G_f(x,z) = e^{ik_x x}G(z)$$

$$\begin{pmatrix} F \\ G \end{pmatrix}_z = \hat{\mathbf{M}} \begin{pmatrix} F \\ G \end{pmatrix}_0$$

ansatz for the field in the cladding:

$$F_c(x,z) = e^{ik_x x}F_T e^{ik_{cz}(z-D)}$$

$$G_c(x,z) = i\alpha_c k_{cz} e^{ik_x x}F_T e^{ik_{cz}(z-D)}$$

Coupling an incident field

$$\begin{pmatrix} F \\ G \end{pmatrix}_D = \hat{\mathbf{M}} \begin{pmatrix} F \\ G \end{pmatrix}_0$$

cladding field at $z = D$

substrate field at $z = 0$

$$\begin{pmatrix} F_T \\ i\alpha_c k_{cz} F_T \end{pmatrix} = \begin{Bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{Bmatrix} \begin{pmatrix} F_I + F_R \\ i\alpha_s k_{sz} \left(F_I - F_R\right) \end{pmatrix}$$

⇨ two equations for the two unknown amplitudes

R/T coefficients of the generalised variables

$$F_R = \frac{(\alpha_s k_{sz} M_{22} - \alpha_c k_{cz} M_{11}) - i\,(M_{21} + \alpha_s k_{sz} \alpha_c k_{cz} M_{12})}{(\alpha_s k_{sz} M_{22} + \alpha_c k_{cz} M_{11}) + i\,(M_{21} - \alpha_s k_{sz} \alpha_c k_{cz} M_{12})} F_I$$

$$F_T = \frac{2\alpha_s k_{sz}\,(M_{11} M_{22} - M_{12} M_{21})}{(\alpha_s k_{sz} M_{22} + \alpha_c k_{cz} M_{11}) + i\,(M_{21} - \alpha_s k_{sz} \alpha_c k_{cz} M_{12})} F_I$$

$\Rightarrow$ absorptionless media $\quad F_T = \dfrac{2\alpha_s k_{sz}}{N} F_I$

$$\| \hat{\mathbf{m}}(x) \| = 1$$

R/T coefficients for TE polarisation

TE polarisation: $F = E = E_y \qquad \alpha_{TE} = 1$

reflected field: $E_R^{TE} = R_{TE} E_I^{TE}$

$$\boxed{R_{TE} = \frac{(k_{sz}M_{22} - k_{cz}M_{11}) - i\,(M_{21} + k_{sz}k_{cz}M_{12})}{(k_{sz}M_{22} + k_{cz}M_{11}) + i\,(M_{21} - k_{sz}k_{cz}M_{12})}}$$

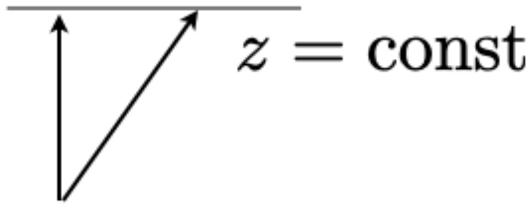
transmitted field: $E_T^{TE} = T_{TE} E_I^{TE}$

$$\boxed{T_{TE} = \frac{2k_{sz}}{N_{TE}}}$$

and similar for TM polarisation: $F = H = H_y \quad \alpha_{TM} = \frac{1}{\epsilon}$

Calculating the efficiencies

- calculating the energy flux
- perpendicular through a surface



$$\langle \mathbf{S} \rangle \cdot \mathbf{e}_z = \frac{1}{2} \Re \left(\mathbf{E} \times \mathbf{H}^* \right) \cdot \mathbf{e}_z$$

using: $$\mathbf{H}^* = \frac{1}{\omega \mu_0} \left(\mathbf{k}^* \times \mathbf{E}^* \right)$$

$$\langle \mathbf{S} \rangle \cdot \mathbf{e}_z = \frac{1}{2\omega\mu_0} \Re \left(\mathbf{k}^* \cdot \mathbf{e}_z \right) \mid \mathbf{E} \mid^2 = \frac{1}{2\omega\mu_0} \Re \left(k_z \right) \mid \mathbf{E} \mid^2$$

$$\rho_{TE,TM} = \mid R_{TE,TM} \mid^2 \quad \tau_{TE,TM} = \frac{\Re \left(k_{cz} \right)}{k_{sz}} \mid T_{TE,TM} \mid^2$$

Theoretical Optics

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