# Theoretische Teilchenphysik I

V: Prof. Kirill Melnikov, Ü: Dr. Matthew Dowling, Dr. Lorenzo Tancredi

## Exercise Sheet 1

Due 22.04.2015

## Problem 1 - One-dimensional chain of springs (100 Points)

In this problem we want to show how the formalism of quantum field theory naturally emerges from a known, discrete, system, once the continuum limit is taken. To this end, we will consider the case of a one-dimensional chain of $N$ springs connected to each other. We will quantize this system for finite values of $N$ and then consider the limit $N \to \infty$, i.e. the continuum limit, see Figure 1.

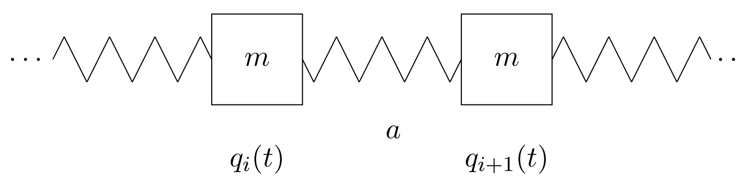


Figure 1: One-dimensional array of springs

1. (20 Points) Let us start by considering a very well known system, the one-dimensional harmonic oscillator. The Lagrangian is given by

$$L = \frac{m}{2}\left(\dot{x}^2 - \omega^2 x^2\right) , \qquad \text{with} \quad \dot{x} = \frac{d\,x}{d\,t} . \tag{1}$$

   a) Derive the expression for the conjugate momentum $p$ and for the Hamiltonian $H$.

   b) Show how the system can be quantized by introducing creation and annihilation operators $a^\dagger, a$ defined as

$$x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right) , \quad p = -i\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right) .$$

   Write the commutation relations between $a, a^\dagger$, check that those commutation relations give correct commutation relations for $x$ and $p$, express the Hamiltonian $H$ in terms of $a, a^\dagger$ and find its eigenvalues.

2. Let us consider now the system in Figure 1. Let us assume that every particle has a mass $m$ and that the displacement of the particle $j$ from its equilibrium position is denoted as $q_j/\sqrt{m}$. We assume that each particle can only move along the line of the array, so that transverse displacements are not allowed. Assuming that every particle interacts only with its two neighbors and that the displacements are small enough (harmonic approximation), the Lagrangian of the system can be written as

$$L = \frac{1}{2}\sum_{j=1}^{N} \dot{q}_j^2 - \frac{\nu^2}{2}\sum_{j=1}^{N-1}\left(q_{j+1} - q_j\right)^2 , \qquad \text{where } \nu \text{ is the stiffness of the springs.} \tag{2}$$

a) (5 Points) Show that, imposing periodic boundary conditions on the system

$$q_{N+1}(t) = q_1(t)\,, \qquad \dot{q}_{N+1}(t) = \dot{q}_1(t)\,,$$

the classical equations of motion are

$$\ddot{q}_j(t) = \nu^2\,[q_{j+1}(t) - 2\,q_j(t) + q_{j-1}(t)]\;, \quad \forall j = 1, N\,. \tag{3}$$

b) (5 Points) We are looking for the solutions that are harmonic functions of time. Such a normal mode solution of Eq. (3) $q_j(t)$ with frequency $\omega$ can be written as

$$q_j(t) = \Re\{A\,e^{-i(K\,j-\omega t)}\}\,, \tag{4}$$

where A is a normalization constant, $\omega$ does not depend on $j$, and $K$ is a constant which can take only a discrete set of values. Show that periodic boundary conditions require

$$K = \frac{2\pi\alpha}{N}\,, \quad \text{with} \quad \alpha = 0, 1, ..., N-1\,.$$

c) (5 Points) Using the exact form of the normal modes together with Eq. (3), show that, for every given value of $\alpha$, the frequency is

$$\omega_\alpha = 2\nu \left|\sin\frac{\pi\alpha}{N}\right|\,,$$

and that the solution for a given normal mode of frequency $\alpha$ can be written as

$$q_j^\alpha(t) = \Re\{a_j^\alpha\,e^{i\omega t}\}\,, \quad \text{with} \quad a_j^\alpha = Ae^{-i2\pi\alpha\,j/N}\,. \tag{5}$$

d) (5 Points) Use the derived expressions for the normal modes to show that the coefficients $a_j^\alpha$ satisfy the orthogonality condition

$$\sum_{j=1}^{N} a_j^{\alpha *} a_j^\beta = C_j\,\delta_{\alpha\beta}\,, \quad \text{where } C_j \text{ is a constant.}$$

Show that imposing the normalization condition $C_j = 1$, i.e.

$$\sum_{j=1}^{N} a_j^{\alpha *} a_j^\beta = \delta_{\alpha\beta}\,,$$

fixes $A = 1/\sqrt{N}$ in Eq. (5).

e) (5 Points) Again, use the derived expressions for the normal modes to show that $a_j^\alpha$ satisfies a completeness relation

$$\sum_{\alpha=0}^{N-1} a_j^{\alpha *} a_k^\alpha = \delta_{jk}\,. \tag{6}$$

f) (10 Points) Given the normal mode solutions $q_j^\alpha(t)$, an arbitrary motion of the particle $j$ will be written as

$$q_j(t) = \Re\left\{\sum_{\alpha=0}^{N-1} c_\alpha\,a_j^\alpha\,e^{i\omega_\alpha\,t}\right\}\,. \tag{7}$$

Show that by introducing the normal coordinates

$$Q_\alpha(t) = \sum_{j=1}^{N} a_j^\alpha q_j(t) \tag{8}$$

the Lagrangian (2) becomes equivalent to the Lagrangian of $N$ independent harmonic oscillators, namely:

$$L = \frac{1}{2}\sum_{\alpha=0}^{N-1}\left(\dot{Q}_\alpha^*\dot{Q}_\alpha - \omega_\alpha^2 Q_\alpha^* Q_\alpha\right)\,. \tag{9}$$

g) (10 Points) Explain how to use Eq. (9) together with the results in point a) in order to quantize the system. What is the commutation relation between $Q_\alpha$ and its conjugate momentum $P_\alpha$ ?

h) (10 Points) Use the completeness relation (6) in order to invert Eq. (8) and derive the expressions for $\hat{q}_j$ and its conjugate momentum $\hat{p}_j$ in terms of the $Q_\alpha$ and $\Pi_\alpha$. We put a hat on $\hat{q}_j$ and $\hat{p}_j$ to recall that they are now operators. Use these results to show that

$$[\hat{p}_j, \hat{q}_k] = i\,\hbar\,\delta_{jk}\,. \tag{10}$$

i) (10 Points) Let us consider now the continuum limit for this system. Given a particular mode of motion the phase difference between adjacent particles is $2\pi\alpha/N$. One wave length will contain $N/\alpha$ particles and if their equilibrium separation is $d$, the wave length can be written as $\lambda = Nd/\alpha$. In the limit when the wave length $\lambda$ is much larger than the relative spacing $d$, one can imagine to describe the system as a continuum. In order to do this, introduce the wave number

$$k = \frac{2\pi}{\lambda}\,,$$

and the equilibrium position of a given particle can be written as $x_j = j\,d$. With these notations we have

$$a_{jk} = \frac{1}{\sqrt{N}} e^{-ikx_j}\,,$$

$$Q_k = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} q_j\, e^{-ikx_j}\,, \quad q_j = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} Q_k e^{ikx_j}\,, \tag{11}$$

and the continuum limit can be taken when $kd \ll 1$. In this limit $q_j$ and $q_{j+1}$ become nearly equal the displacement can be seen as a continuous function of the position $x$ on the line, namely

$$q_j = q(x_j) \approx \sqrt{m}\,\phi(x)\,,$$

where the $\sqrt{m}$ is needed to account for the normalization we have chosen to define the $q_j$ in (2). Using the formal replacement

$$\sum_{j=1}^{N} (...) \to \frac{N}{L}\int_0^L (...)dx\,, \quad \text{where } L \text{ is the length of the string}\,,$$

show that the Lagrangian in Eq. (2) can be written as

$$L = \frac{\rho}{2}\int_0^L \left(\frac{\partial\phi}{\partial t}\right)^2 dx - \frac{\rho c^2}{2}\int_0^L \left(\frac{\partial\phi}{\partial x}\right)^2 dx = \int_0^L \left\{\frac{\rho}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{\rho c^2}{2}\left(\frac{\partial\phi}{\partial x}\right)^2\right\} dx\,, \tag{12}$$

where we introduced the mass density $\rho = m/d$ and the constant $c = \nu\, d$.

j) (10 Points) Write down the Euler-Lagrange equation for the field $\phi$ that follows from the Lagrangian Eq.(12) and use it to give a physical interpretation of the constant $c$.

k) (5 Points) Suppose you want to quantize the continuous system described by the Lagrangian Eq.(12). Given the relation between the field $\phi(x_i)$ and the original discrete coordinate $q_i$ described in part i), and the standard discrete quantization condition Eq. 10, can you guess the quantization condition for the field $\phi(x)$ and its conjugate momentum $\pi(x)$ that needs to be applied in the continuum limit?