Theoretische Teilchenphysik I

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Exercise Sheet 10

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Problem 1 - One-loop QED vacuum polarization tensor (100 points)

In this problem we will calculate the one-loop corrections to the vacuum polarization tensor in QED. We start off by considering the photon polarization function $\Pi^{\mu\nu}(q^2)$ in total generality. The latter is a rank-two tensor and therefore, because of Lorentz invariance, it can only be of the form

$$\Pi^{\mu\nu}(d, q^2) = A(d, q^2)\, q^2\, g^{\mu\nu} + B(d, q^2)\, q^\mu q^\nu\,, \tag{1}$$

where $A(d, q^2)$ and $B(d, q^2)$ must be *scalar functions* of q^2 and of the dimensional regularization parameter d.

1. (5 points) Show that, by imposing gauge invariance, i.e. $q_\mu \Pi^{\mu\nu}(d, q^2) = q_\nu \Pi^{\mu\nu}(d, q^2) = 0$, one must have
$$B(d, p^2) = -A(d, p^2)\,,$$
such that
$$\Pi^{\mu\nu}(d, q^2) = A(d, q^2)\,\left(q^2\, g^{\mu\nu} - q^\mu q^\nu\right)\,. \tag{2}$$

2. (10 points) Show that equation (2) can be inverted to give
$$A(d, p^2) = \frac{1}{q^2(d-1)}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Pi_{\mu\nu}(d, q^2)$$
$$= P^{\mu\nu}(d, q^2)\,\Pi_{\mu\nu}(d, q^2)\,, \tag{3}$$
where we defined the *projector*
$$P^{\mu\nu}(d, q^2) = \frac{1}{q^2(d-1)}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\,, \tag{4}$$
What we derived until know is independent on the perturbative expansion in Feynman diagrams, but remains of course true if we consider the perturbative expression for the polarization tensor.

3. (10 points) Let us consider now the QED one-loop corrections to $\Pi^{\mu\nu}(d, q^2)$. First of all, draw the Feynman diagram contributing at one-loop order and show that it's contribution to the photon polarization tensor reads
$$\Pi^{\mu\nu}(d, q^2) = e^2 \int \frac{d^d k}{(2\pi)^d}\,\mathrm{Tr}\left[\gamma^\mu \frac{1}{\not{k} - m}\gamma^\nu \frac{1}{\not{k} + \not{q} - m}\right]\,, \tag{5}$$
where e is the electron charge.

4. (15 points) Evaluate the trace in Eq. (5) and use the projector in Eq. (4) in order to extract the one-loop corrections to the coeffcient $A(d,p^2)$. Show that

$$A(d,p^2) = \frac{2\,e^2}{(d-1)q^2}\left\{2(d-2)\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)}\right.$$
$$\left.-\left[q^2(d-2)+4\,m^2\right]\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)((k+q)^2-m^2)}\right\}. \tag{6}$$

5. (15 points) Prove that

$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)} = -i\,\frac{C_d}{16\pi^2}\,\frac{4\,m^2}{(d-2)(d-4)}\,,$$
$$\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m^2)((k+q)^2-m^2)} = -i\,\frac{C_d}{16\pi^2}\,\frac{2}{(d-4)}\int_0^1 dx\left(1-\frac{q^2}{m^2}x(1-x)\right)^{(d-4)/2}\,, \tag{7}$$

where

$$C_d = \frac{\Gamma\left(\frac{6-d}{2}\right)\,\left(m^2\right)^{(d-4)/2}}{(4\pi)^{(d-4)/2}}\,.$$

6. (10 points) The expression derived in Eq. (6) has an apparent pole as $q^2 \to 0$. Using the expressions derived in Eqs. (7), show that the function $A(d,p^2)$ is regular as $q^2 \to 0$, and that its value is

$$A(d,0) = i\left(\frac{2\alpha}{\pi}\right)\frac{1}{3}\frac{C_d}{(d-4)}\,. \tag{8}$$

7. (15 points) Plugging Eq. (7) into Eq. (6) and expanding up to first order in $(d-4)$, show that

$$A(d\to 4,p^2) = i\left(\frac{2\alpha}{\pi}\right)C_d\left[\frac{1}{3(d-4)}+\frac{1}{18}+\frac{(q^2+2\,m^2)}{6\,q^2}\int_0^1 dx\,\ln\left(1-\frac{q^2}{m^2}\,x(1-x)\right)\right]+\mathcal{O}(d-4)\,, \tag{9}$$

where note that the divergent piece in Eq. (9) is identical to Eq. (8).

8. (15 points) The divergence in Eq. (9) must be removed by *UV renormalization*. A reasonable prescription consists in imposing that the renormalized polarization tensor is zero for $q^2 \to 0$, i.e.

$$\Pi^{\mu\nu}_{ren}(q^2\to 0)\to 0\,, \tag{10}$$

which can be achieved defining

$$A_{ren}(d,p^2) = A(d,p^2) - A(d,0)\,.$$

What does this prescription mean physically?