# Theoretische Teilchenphysik I

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## Exercise Sheet 4

Due 13.05.2015

## Problem 1 - Lorentz transformations 1/3 (25 points)

Lorenz transformations can be thought of as $4 \times 4$ matrices $\Lambda^{\mu}_{\nu}$ acting on four-vectors by matrix multiplications. Defining $x^{\mu} = (t, \vec{x})$

$$x'^{\mu} = \Lambda^{\mu}_{\nu}\, x^{\nu}\,.$$

Consider $\Lambda$'s that are infinitesimally close to the identity matrix $\Lambda^{\mu}_{\nu} \approx \delta^{\mu}_{\nu} + \epsilon\, A^{\mu}_{\nu}$, with $0 < \epsilon \ll 1$ .

a) (5 points) What condition should the matrix $A$ satisfy to make $\Lambda$ a Lorenz transformation?

b) (5 points) Find the form of $A$ that corresponds to rotations around the $i$-th axis. Use this to write an explicit matrix for a rotation around the z-axis.

c) (5 points) Find the form of $A$ that corresponds to boosts along the $i$-th axis. Use this to write an explicit matrix for a boost in the x-direction.

d) (10 points) Show that any $A$ that satisfies the conditions derived in point $a$) can be written as a linear combination of boosts and rotations.

## Problem 2 - Lorentz transformations 2/3 (25 points)

Consider a scalar quantum field $\phi(x)$. Show that the transformation rule

$$U(\Lambda)^{-1}\phi(x)U(\Lambda) = \phi(\Lambda^{-1}x) \tag{1}$$

implies the following commutation relation of the field $\phi$ and the generators of the Lorentz group

$$[\mathcal{M}^{\mu\nu}, \phi(x)] = J^{\mu\nu}\phi(x), \tag{2}$$

where $J^{\mu\nu} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu})$.

## Problem 3 - Lorentz transformations 3/3 (20 points)

As discussed in class, generators of the Lie algebra of the Lorentz group satisfy the following commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\rho}M^{\sigma\nu} - g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\rho\nu} + g^{\nu\rho}M^{\mu\sigma}\right). \tag{3}$$

There are six independent generators $M^{\mu\nu}$. We take them to be three rotations $J_i = 1/2\epsilon_{ijk}M^{jk}$ and three boosts $K_i = M^{i0}$. Show that these operators satisfy the following commutation relations

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \quad [K_i, K_j] = -i\epsilon_{ijk}J_k, \quad [J_i, K_j] = i\epsilon_{ijk}K_k. \tag{4}$$

## Problem 4 - Dirac gamma matrices (30 points)

Following the discussion in class, show by an explicit computation that

$$e^{i\vec{\phi}\vec{K}}\gamma_0 e^{-i\vec{\phi}\vec{K}} = \frac{\gamma^\mu p_\mu}{m}, \tag{5}$$

where $\gamma^\mu$ are Dirac matrices in the Weyl representation.
Remember: $\cosh(\phi) = \frac{E}{m}$ and $\vec{\phi} = \frac{\vec{p}}{|\vec{p}|}\phi$.

Generalize this result to show that, for boosts with $\Lambda_{1/2} = e^{-i\vec{\phi}\cdot\vec{K}}$

$$\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu_\nu\gamma^\nu. \tag{6}$$