Theoretische Teilchenphysik I

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Exercise Sheet 5

Due 20.05.2015

Problem 1 - Majorana fermions (100 points)

The Majorana equation for the two component fermion field reads

$$\bar{\sigma}_\mu \partial^\mu \Psi_L + m\sigma_2 \Psi_L^* = 0\,, \tag{1}$$

where we used the γ matrices in the Weyl representation and $\bar{\sigma} = (1, -\sigma_j)$. In this problem we will find an explicit representation of the Majorana spinor Ψ_L using plane wave solutions.

a) (20 points)To start, show that the Majorana field defined in class produces Eq. 1. Remember, the field is defined by

$$\Psi(x) = \begin{pmatrix} \Psi_L(x) \\ -i\sigma_2 \Psi_L^*(x) \end{pmatrix}, \tag{2}$$

where Ψ_L has two components.

b) (20 points) We want a momentum representation of the field. This means that our solution will be written in the form $Ae^{\pm ip\cdot x}$. To start, derive the relation

$$\overline{\sigma}^\mu \partial_\mu e^{\pm ip\cdot x} = \pm i \left(E + \vec{\sigma}\cdot\vec{p}\right) e^{\pm ip\cdot x}. \tag{3}$$

c) (30 points)The Majorana fermion is massive so it does not have a Lorentz invariant helicity. This means that we must include both helicities in the momentum representation of the field. The most general form of a solution is then

$$\Psi_L(x) = \sum_{r=1,2} \int_p \left(a_r(p)\zeta_r(p) e^{-ip\cdot x} + a_r^\dagger(p)\eta_r(p)e^{+ip\cdot x}\right)\,. \tag{4}$$

Use this in Eq. 1 to derive the relations

$$\eta_r = \frac{E - \vec{\sigma}\cdot\vec{p}}{m} i\sigma_2 \zeta_r^*, \tag{5}$$

by equating the coefficients of a_1 and a_2.

d) (30 points) From here, any linear independent choice of the ζ_r's is a solution of the 2-component Majorana equation. As an example, use

$$\zeta_1 = \xi_-, \quad \eta_2 = \xi_-, \tag{6}$$

where, $\xi_\pm$ are helicity eigenvectors satisfying the relation

$$\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|}\xi_\pm(p) = \pm\xi_\pm(p), \tag{7}$$

and fulfil the relation $\xi_- = i\,\sigma_2 \xi_+^*$, to show that the Fourier expansion of $\Psi_L(x)$ can be written as

$$\begin{aligned}\Psi_L(x) \quad = \quad \int_p \Bigg(& a_-(p)\xi_-(p)e^{-ip\cdot x} + \frac{m}{E+|\vec{p}|} a_+(p)\xi_+(p)e^{-ip\cdot x} \\ & + a_+^\dagger(p)\xi_-(p)e^{+ip\cdot x} + \frac{m}{E+|\vec{p}|} a_-^\dagger(p)\xi_+(p)e^{+ip\cdot x}\Bigg).\end{aligned} \tag{8}$$