Theoretische Teilchenphysik I

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Exercise Sheet 6

Due 27.05.2015

Problem 1 - Nöther's Theorem (40 points)

Nöther's Theorem says that any continuous global symmetry of the action leads to a conserved current for solutions of the equations of motion. In this problem we will derive the energy-momentum tensor of a scalar field, ϕ, using Nöther's Theorem.

a) (5 points) In classical mechanics invariance under spatial (time) translations gives rise to conservation of momentum (energy). This relation holds in classical field theory where space-time translations result in conservation of the energy-momentum tensor.

   Consider a theory with the Lagrange density $\mathcal{L}(\phi(x), \partial_\mu \phi(x))$. Show that the action $S = \int \mathrm{d}^4x \mathcal{L}$ is invariant under space-time translations

$$x^\mu = y^\mu - \epsilon^\mu. \tag{1}$$

b) (5 points) How do the scalar field $\phi(x)$ and the Lagrangian $\mathcal{L}$ transform as a result of this translation?

c) (10 points) Using Nöther's Theorem, show that the conserved quantity related to space-time translations is,

$$(T^\mu)^\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \equiv T^{\mu\nu} \tag{2}$$

   This tensor is the stress-energy (or energy-momentum) tensor.

d) (10 points) Calculate explicitly the energy-momentum tensor in a theory with the Lagrange-density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 + V(\phi). \tag{3}$$

   Write down T^{00} and compare it with the Hamiltonian of the problem.

e) (10 points) Conservation of the energy-momentum tensor implies

$$\partial_\mu T^{0\mu} = 0. \tag{4}$$

   Integrate this equation over the finite volume and explain the physical meaning of various terms that appear.

1 Problem 2 - Discrete Symmetries (20 points)

Nöther's Theorem applies only to continuous symmetries. There exist, as well, discrete symmetries that can restrict the types of terms appearing in the Lagrangian. In this problem we will explore a few of these symmetries and how they provide information about the Lagrangian.

a) (10 points) Suppose the Lagrangian of the real world is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 + a\phi + b\phi^2 + c\phi^3 + d\phi^4 \tag{5}$$

for ϕ real. The real world is symmetric under the discrete transformation $\phi \to -\phi$ (so called Z_2-symmetry). What can you say about the coefficients a, b, c, d in the Lagrangian?

b) (10 points) The Lagrangian describing fermions interacting with photons is

$$\mathcal{L} = \overline{\psi}(i\hat{\partial} - m)\psi + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + e_1\overline{\psi}\gamma_\mu\psi A^\mu + e_2\overline{\psi}\gamma_\mu\gamma^5\psi A^\mu, \tag{6}$$

where ψ is a fermion field and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. If A^μ is a vector under parity and partity is a symmetry of the lagrangian, what can you say about the coefficients e_1 and e_2.

2 Problem 3 - Conserved charge of the complex-scalar theory (40 points)

Consider a theory for a free complex scalar field described by the Lagrangian density

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2\, \phi^* \phi\,. \tag{7}$$

1. (5 points) Use Nöther theorem to prove that the conserved current can be written as

$$j^\mu(t, \vec{x}) = -i\, \left(\phi\, (\partial^\mu \phi)^* - \phi^*\, (\partial^\mu \phi)\right)\,, \tag{8}$$

note that the overall normalization constant is arbitrary.

2. (15 points) From the previous exercises you know that the quantized complex scalar field can be written in terms of two sets of creation and annihilation operators as

$$\phi(t, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2\omega_p}} \left(a_p\, e^{-ip\cdot x} + b_p^\dagger\, e^{ip\cdot x}\right)\,. \tag{9}$$

Write the current j^μ in terms of the creation and annihilation operators $a_p, a_p^\dagger, b_p, b_p^\dagger$.

3. (10 points) Finally, compute the conserved charge

$$Q = \int d^3\vec{x}\, j^0(t, \vec{x}) \tag{10}$$

in terms of $a_p, a_p^\dagger, b_p, b_p^\dagger$.

4. (10 points) Does the operator Q commute with the Hamiltonian? Evaluate the value of the charge of the particles of the two different types, namely

$$Q\, a_p^\dagger|0\rangle = \dots\,, \qquad Q\, b_p^\dagger|0\rangle = \dots\,.$$

Can you give a physical interpretation of the operator Q?