Theoretische Teilchenphysik I

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Exercise Sheet 8

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Problem 1 - Kinematics of $2 \to 2$ processes (60 points)

In this problem we want to study the kinematics of a $2 \to 2$ scattering process in terms of the so-called Mandelstam variables s, t, u.

1. We start with a simple case and consider a generic process which involves the interaction of 4 massless particles of momenta p_1, p_2, p_3, p_4 with $p_j^2 = 0$. To keep the discussion general we consider all four particles to be incoming (see figure). This process can be written as

$$p_1 + p_2 + p_3 + p_4 \to 0\,. \tag{1}$$

Written in this way, the process is non-physical, in the sense that it cannot happen in reality, but we will use it to establish the notation. We define the Mandelstam variables as

$$s = (p_1 + p_2)^2\,, \quad t = (p_1 + p_3)^2\,, \quad u = (p_1 + p_4)^2 = (p_2 + p_3)^2\,. \tag{2}$$

1a) (5 points) Explain why the Mandelstam variables can be computed in *any* reference frame and prove that

$$s + t + u = 0\,.$$

1b) (10 points) Consider a physical process where particles p_1 and p_2 produce particles p_3 and p_4

$$p_1 + p_2 \to p_3 + p_4\,. \tag{3}$$

Since the incoming total momentum is $(p_1 + p_2)^2 = s$, this is called the *s-channel process*.
Go to the center-of-mass frame and introduce the scattering angle θ defined as

$$\vec{p}_1 \cdot \vec{p}_3 = |\vec{p}_1||\vec{p}_3| \cos\theta\,,$$

and the total energy $p_1^0 + p_2^0 = W$. Express s, t, u in terms of W and θ and prove that for the process (3) one always has

$$s > 0\,, \quad t < 0\,, \quad u < 0\,. \tag{4}$$

1c) (5 points) Draw the physical region allowed for the process in the $s - t$ plane. Note that for all allowed values of s and t, one must also have $u < 0$. This type of plot is referred to as the *Dalitz plot* for a particular process.

1d) (10 points) Consider now the related process in the t-channel. In this case, p_1 and p_3 produce p_2 and p_4

$$p_1 + p_3 \to p_2 + p_4\,. \tag{5}$$

Draw the physical region allowed for this process in the same $s - t$ plane of point 1c).

2. We generalise the preceeding question and consider the process

$$p_1 + p_2 + p_3 + p_4 \to 0\,,$$

where now two particles, say 3 and 4, are massive, i.e. $p_1^2 = p_2^2 = 0\,,\ p_3^2 = p_4^2 = m^2$.

2a) (5 points) Introduce the three Mandelstam variables and prove that in this case

$$s + t + u = 2\,m^2\,.$$

2b) (10 points) Study the physical s-channel process

$$p_1 + p_2 \to p_3 + p_4$$

in the center-of-mass frame by introducing the scattering angle θ and the total center-of-mass energy W. What are the physically allowed values for θ and W?

2c) (10 points) Express s, t, u in terms of W, θ and m^2. What are the allowed values for s, t and u? Draw the physical region for this process in the $s - t$ plane. Recall again that the values allowed for s, t are constrained also from the values of $u = 2m^2 - s - t$.

2d) (5 points) Similar to the massless case, consider the t-channel process

$$p_1 + p_3 \to p_2 + p_4 \tag{6}$$

and draw the physical region allowed for this process on the same $s - t$ plane of point 2c).

Problem 2 - Iterative construction of 3-particle phase space (40 points)

In the following exercise we want to calculate the 3-body phase space *iteratively* starting from the 2-body phase space.

0. (5 points) Prove that
$$\int \frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} = \int \frac{d^4p}{(2\pi)^4}(2\pi)\theta(p_0)\,\delta(p^2 - m^2)\,. \tag{7}$$
where $\omega_p = \sqrt{\vec{p}^2 + m^2}$,

1. (10 points) Start by considering a system of two particles of masses m_1 and m_2, total 4-momentum $P^\mu = k_1^\mu + k_2^\mu$ and the total center-of-mass energy W, with $P^\mu P_\mu = s = W^2$.

   The 2-body phase space of this system is defined as
$$\Phi_2(s, m_1^2, m_2^2) = \int \frac{d^3\vec{k}_1}{(2\pi)^3 2\omega_{k_1}} \int \frac{d^3\vec{k}_2}{(2\pi)^3 2\omega_{k_2}} (2\pi)^4\,\delta^{(4)}(P - k_1 - k_2)\,. \tag{8}$$
   Go to the center-of-mass frame and use Eq. (7) to show that
$$\Phi_2(s, m_1^2, m_2^2) = \frac{1}{8\pi\,s}\sqrt{\lambda(s, m_1^2, m_2^2)}\,, \tag{9}$$
   where $\lambda(a, b, c)$ is called *Källen function* and is defined as
$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2\,a\,b - 2\,a\,c - 2\,b\,c\,.$$

2. Consider now a system of 3 particles of masses m_1, m_2 and m_3, total momentum $P^\mu = (k_1 + k_2 + k_3)^\mu$ and total center-of-mass energy $W = \sqrt{P_\mu P^\mu} = \sqrt{s}$. The 3-body phase space is defined as
$$\Phi_3(s, m_1^2, m_2^2, m_3^2) = \int \frac{d^3\vec{k}_1}{(2\pi)^3 2\omega_{k_1}} \int \frac{d^3\vec{k}_2}{(2\pi)^3 2\omega_{k_2}} \int \frac{d^3\vec{k}_3}{(2\pi)^3 2\omega_{k_3}} (2\pi)^4 \delta^4(P - k_1 - k_2 - k_3)\,. \tag{10}$$
   We want to evaluate it iteratively starting from the result in Eq. (9).

2a) (5 points) Start by considering the two particles 2 and 3 with masses m_2 and m_3. Suppose their total momentum is $q = k_2 + k_3$ and their invariant mass is $q^2 = M^2$. Prove that the physically allowed values of M^2 are $(m_2 + m_3)^2 \leq M^2 \leq (\sqrt{s} - m_1)^2$.

2b) (5 points) Prove that

$$\int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} \frac{dM^2}{2\pi} \int \frac{d^4q}{(2\pi)^4} (2\pi)\theta(q_0)\delta(q^2 - M^2)(2\pi)^4\delta^4(q - k_2 - k_3) = 1\,. \tag{11}$$

2c) (10 points) Multiply the definition of the 3-body phase space, Eq. (10), by the left hand side of Eq. (11), and show that

$$\begin{aligned}\Phi_3(s, m_1^2, m_2^2, m_3^2) &= \int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} \frac{dM^2}{2\pi} \Phi_2(s, M^2, m_1^2)\Phi_2(M^2, m_2^2, m_3^2) \\ &= \frac{1}{8\pi s} \int_{(m_2+m_3)^2}^{(\sqrt{s}-m_1)^2} \frac{dM^2}{2\pi} \frac{1}{8\pi M^2} \sqrt{\lambda(s, M^2, m_1^2)\lambda(M^2, m_2^2, m_3^2)}\,. \end{aligned} \tag{12}$$

The result in Eq. (12) shows that the 3-body phase space can be seen as the product of the 2-body phase space of particles m_2, m_3, times the 2-body phase space of particle m_1 together with the system of the other two particles of invariant mass M^2, integrated over the physically allowed values of M^2.

2d) (5 points) Show that in case of two masless and one massive particle, e.g.

$$m_1 = m_2 = 0\,, \qquad \text{and} \qquad m_3 = m\,,$$

the result for the three-particle phase-space Eq. (12) simplifies dramatically. Explicitly compute $\Phi_3(s, 0, 0, m^2)$ in this case.