Theoretische Teilchenphysik I

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Exercise Sheet 9

Due 17.06.2015

Problem 1 - Fierz Transformations (60 points)

In this exercise we will study in detail some properties of the Dirac matrices γ^μ.

a) (15 points) We start considering traces of γ-matrices. Using the cyclicity of the trace, namely $\mathrm{tr}(A\,B) = \mathrm{tr}(B\,A)$, prove that

$$\mathrm{tr}(\gamma^\mu \gamma^\nu) = 4\, g^{\mu\nu}\,, \tag{1}$$

$$\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 0\,, \tag{2}$$

$$\mathrm{tr}(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})\,, \tag{3}$$

$$\mathrm{tr}(\gamma^{\mu_1} \gamma^{\mu_2} ... \gamma^{\mu_n}) = 0\,, \quad \forall\, n\, \mathrm{odd}. \tag{4}$$

Now, any 4×4 constant matrix Γ can be written in terms of 16 combinations of Dirac gamma matrices,

$$\Gamma^a = \{1, \gamma^\mu, \sigma^{\mu\nu}, \gamma^\mu \gamma^5, \gamma^5\}. \tag{5}$$

This basis allows us to write identities such as,

$$\overline{u}_1 \gamma^\mu \left(\frac{1+\gamma^5}{2}\right) u_2 \overline{u}_3 \gamma_\mu \left(\frac{1+\gamma^5}{2}\right) u_4 = -\overline{u}_1 \gamma^\mu \left(\frac{1+\gamma^5}{2}\right) u_4 \overline{u}_3 \gamma_\mu \left(\frac{1+\gamma^5}{2}\right) u_2. \tag{6}$$

These identities are known as Fierz transformations and there are similar formulae for any product

$$(\overline{u}_1 \Gamma^A u_2)(\overline{u}_3 \Gamma^B u_4), \tag{7}$$

where Γ^A, Γ^B are any of the matrices from the set Γ^a.

b) (10 points) To begin, normalize the 16 matrices to match the convention

$$tr[\Gamma^A \Gamma^B] = 4\delta^{AB}, \tag{8}$$

and write all 16 normalized elements.

c) (20 points) We write the general Fierz identity as an equation

$$(\overline{u}_1 \Gamma^A u_2)(\overline{u}_3 \Gamma^B u_4) = \sum_{C,D} C^{AB}_{CD} (\overline{u}_1 \Gamma^C u_4)(\overline{u}_3 \Gamma^D u_2), \tag{9}$$

with unknown coefficients C^{AB}_{CD}. Using the completeness of the 16 basis matrices, show that

$$C^{AB}_{CD} = \frac{1}{16} tr[\Gamma^C \Gamma^A \Gamma^D \Gamma^B]. \tag{10}$$

d) (15 points) Work out explicitly the Fierz transformation laws for the products $(\overline{u}_1 u_2)(\overline{u}_3 u_4)$ and $(\overline{u}_1 \gamma^\mu u_2)(\overline{u}_3 \gamma_\mu u_4)$.

Problem 2 - The decay of a scalar particle (40 points)

Consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}M^2\Phi^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2 - \mu\,\Phi\,\phi\,\phi\,, \tag{11}$$

which involves two scalar fields Φ and ϕ of masses respectively M and m, and which can interact with each other through the interaction Lagrangian $\mathcal{L}_{int} = -\mu\,\Phi\,\phi\,\phi$, where μ is the coupling constant. If $M > 2\,m$, this interaction term allows a particle of type Φ to decay into two particles of type ϕ.

a) (10 points) Using the techniques discussed in class, derive the Feynman rules for this theory.

b) (15 points) Consider the decay $\Phi \to \phi + \phi$. Draw the Feynman diagram contributing to this process at zeroth-order in perturbation theory (i.e., tree-level). Use it to derive the expression for the amplitude squared $|\mathcal{M}|^2_{\Phi\to\phi\phi}$.

c) (15 points) Use the result in point b) to compute the lifetime of the particle Φ to lowest order in μ.