QFT – 2

Problem 1

a) $$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{1}{2} m^2 \phi_1^2 - \frac{1}{2} m^2 \phi_2^2$$

$$\frac{\partial \mathcal{L}}{\partial \phi_1} = -m^2 \phi_1 \qquad \frac{\partial \mathcal{L}}{\partial \phi_2} = -m^2 \phi_2$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} = \partial_\mu (\partial^\mu \phi_1) = \partial^2 \phi_1$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} = \partial_\mu (\partial^\mu \phi_2) = \partial^2 \phi_2$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial \phi} \Rightarrow (\partial^2 + m^2)\phi_1 = 0,$$
$$(\partial^2 + m^2)\phi_2 = 0.$$

$$\pi_1 = \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_1)} = \partial^0 \phi_1$$
$$\pi_2 = \frac{\partial \mathcal{L}}{\partial(\partial^0 \phi_2)} = \partial^0 \phi_2$$

$$\mathcal{H} = \sum_i \pi_i \partial^0 \phi_i - \mathcal{L}$$
$$= (\partial^0 \phi_1)^2 + (\partial^0 \phi_2)^2 - \frac{1}{2}(\partial^0 \phi_1)^2 - \frac{1}{2}(\partial^0 \phi_2)^2 + \frac{1}{2}(\vec{\nabla}\phi_1)^2 + \frac{1}{2}(\vec{\nabla}\phi_2)^2 + \frac{1}{2}m^2\phi_1^2 + \frac{1}{2}m^2\phi_2^2$$

$$= \frac{1}{2}\left((\partial_t \phi_1)^2 + (\partial_t \phi_2)^2 + (\vec{\nabla}\phi_1)^2 + (\vec{\nabla}\phi_2)^2 + m^2\phi_1^2 + m^2\phi_2^2 \right)$$

$$= \frac{1}{2}\left[\sum_i \left(\pi_i^2 + (\vec{\nabla}\phi_i)^2 + m^2\phi_i^2\right) \right]$$

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b) $[\phi_i(t,\vec{x}), \phi_j(t,\vec{y})] = [\pi_i(t,\vec{x}), \pi_j(t,\vec{y})] = 0$

$[\phi_i(t,\vec{x}), \pi_j(t,\vec{y})] = i\,\delta_{ij}\,\delta(\vec{x}-\vec{y})$

$i,j = 1,2.$

let $[a_{\vec{p}}, a^{\dagger}_{\vec{p}'}] = [b_{\vec{p}}, b^{\dagger}_{\vec{p}'}] = (2\pi)^3\delta(\vec{p}-\vec{p}')$

and $[a_{\vec{p}}, b_{\vec{p}'}] = 0 = [a_{\vec{p}}, a_{\vec{p}'}]$
$= [b_{\vec{p}}, b_{\vec{p}'}] = [a^{\dagger}_{\vec{p}}, a^{\dagger}_{\vec{p}'}] =$
$[b^{\dagger}_{\vec{p}}, b^{\dagger}_{\vec{p}'}] = [a^{\dagger}_{\vec{p}}, b^{\dagger}_{\vec{p}'}]$

and $[a_{\vec{p}}, a^{\dagger}_{\vec{p}'}] = [b_{\vec{p}}, b^{\dagger}_{\vec{p}'}] = (2\pi)^3 \quad \delta(\vec{p}-\vec{p}')$

$$\Rightarrow \phi_1(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{ a_{\vec{p}}\, e^{+ip_\mu x^\mu} + a^{\dagger}_{\vec{p}}\, e^{-ip_\mu x^\mu}\right\}$$

$$\phi_2(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{ b_{\vec{p}}\, e^{+ip_\mu x^\mu} + b^{\dagger}_{\vec{p}}\, e^{-ip_\mu x^\mu}\right\}$$

convention

then

$$\pi_1(t,\vec{x}) = \partial_t \phi_1(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{-iE_p}{\sqrt{2E_p}}\left\{ a_{\vec{p}}\, e^{-ip_\mu x^\mu} - a^{\dagger}_{\vec{p}}\, e^{ip_\mu x^\mu}\right\}$$

$$\pi_1(t,\vec{x}) = \int \frac{d^3p}{\sqrt{2\pi}^3}(-i)\sqrt{\frac{\omega_p}{2}}\left\{ a_{\vec{p}}\, e^{-ip_\mu x^\mu} - a^{\dagger}_{\vec{p}}\, e^{+ip_\mu x^\mu}\right\}$$

$$\pi_2(t,\vec{x}) = \int \frac{d^3p}{\sqrt{2\pi}^3}(-i)\sqrt{\frac{\omega_p}{2}}\left\{ b_{\vec{p}}\, e^{-ip_\mu x^\mu} - b^{\dagger}_{\vec{p}}\, e^{ip_\mu x^\mu}\right\}$$

and thus trivially $[\phi_i(t,\vec{x}), \phi_j(t,\vec{y})] =$
$[\pi_i(t,\vec{x}), \pi_j(t,\vec{y})] = [\phi_i(t,\vec{x}), \pi_j(t,\vec{y})] = 0$ for $i \neq j$.

We then find $[\phi_i(t,\vec{x}), \phi_i(t,\vec{y})]$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2\omega_p}} \frac{1}{\sqrt{2\omega_{p'}}} \left[a_{\vec{p}} e^{-ipx} + a^\dagger_{\vec{p}} e^{ipx}, a_{\vec{p}'} e^{-ip'y} + a^\dagger_{\vec{p}'} e^{ip'y}\right]$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{4\omega_p \omega_{p'}}} (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}\,') \times \left[e^{-ipx} e^{ip'y} - e^{ipx} e^{-ip'y}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(e^{-ip(x-y)} - e^{ip(x-y)}\right) \frac{1}{2\omega_p}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(e^{+i\vec{p}(\vec{x}-\vec{y})} - e^{-i\vec{p}(\vec{x}-\vec{y})}\right) \frac{1}{2\omega_p}$$

$$\propto \delta^{(3)}(\vec{x}-\vec{y}) - \delta^{(3)}(\vec{x}-\vec{y}) = 0$$

and anagolously $[\pi_i(t,\vec{x}), \pi_i(t,\vec{y})] = 0$.

Also $[\phi_i(t,\vec{x}), \pi_j(t,\vec{y})]$

$$= -\frac{i}{2} \int \frac{d^3p\, d^3p'}{(2\pi)^6} \sqrt{\frac{\omega_{p'}}{\omega_p}} \left[a^i_{\vec{p}} e^{-ipx} + a^{i\dagger}_{\vec{p}} e^{ipx}, a^j_{\vec{p}'} e^{-ip'y} - a^{j\dagger}_{\vec{p}'} e^{ip'y}\right]$$

$$= -\frac{i}{2} \int \frac{d^3p\, d^3p'}{(2\pi)^6} \sqrt{\frac{\omega_{p'}}{\omega_p}} \left(-e^{-ipx} e^{ip'y} - e^{ipx} e^{-ip'y}\right) \delta(\vec{p}-\vec{p}\,')(2\pi)^3 \delta_{ij}$$

$$= \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} \left(e^{ip(x-y)} + e^{ip(y-x)}\right) \delta_{ij}$$

$$= i\, \delta^{(3)}(\vec{x}-\vec{y})\, \delta_{ij} \quad \text{as wished.}$$

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(writing $a^{(\dagger)}_{\vec{p}} =: a^{1(\dagger)}_{\vec{p}}$, $b^{(\dagger)}_{\vec{p}} = a^{2(\dagger)}_{\vec{p}}$).

c) ~~$\mathcal{H}=$~~

$$H = \int d^3x \; \mathcal{H}$$

$$\mathcal{H} = \cancel{\sum_{i=1}^{2} \frac{1}{2}\left[(\partial_t \phi_i)^2 + (\vec{\nabla}\phi_i)^2 + m^2 \phi_i^2\right]}$$

$$= \frac{1}{2}\sum_{i=1}^{2} \left(\pi_i^2 + (\vec{\nabla}\phi_i)^2 + m^2 \phi_i^2 \right)$$

$$\Rightarrow \quad H = \frac{1}{2}\int d^3x \sum \pi_i^2 + (\vec{\nabla}\phi_i)^2 + m^2\phi_i^2$$

$$= \sum_{i=1}^{2} \frac{1}{2}\int d^3x \left[\pi_i^2 + (\vec{\nabla}\phi_i)^2 + m^2\phi_i^2\right]$$

this is exactly the exercise here! $\longrightarrow$

$$= \sum_{i=1}^{2} \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \left(a^{i\dagger}_{\vec{p}} a^{i}_{\vec{p}} + \frac{1}{2}\left[a^{i}_{\vec{p}}, a^{i\dagger}_{\vec{p}}\right]\right)$$

from the lecture (tedious calculation but simple)

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3} \Big(a^\dagger_{\vec{p}} a_{\vec{p}} + b^\dagger_{\vec{p}} b_{\vec{p}} + \frac{1}{2}\left[a_{\vec{p}}, a^\dagger_{\vec{p}}\right] + \frac{1}{2}\left[b_{\vec{p}}, b^\dagger_{\vec{p}}\right] \Big)$$

■

also seen in problem 2!

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d) $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2} m \phi_1^2 - \frac{1}{2} m^2 \phi_2^2$

$$= \left(\partial_\mu \frac{\phi_1}{\sqrt{2}}\right)\left(\partial^\mu \frac{\phi_1}{\sqrt{2}}\right) + \left(\partial_\mu \left(\frac{-i\phi_2}{\sqrt{2}}\right)\right)\left(\partial^\mu \frac{i\phi_2}{\sqrt{2}}\right)$$

$$- m^2 \left(\left(\frac{\phi_1}{\sqrt{2}}\right)^2 + \left(\frac{-i\phi_2}{\sqrt{2}}\right)\left(\frac{i\phi_2}{\sqrt{2}}\right)\right)$$

$$= \left(\partial_\mu \left(\frac{\phi_1 - i\phi_2}{\sqrt{2}}\right)\right)\left(\partial^\mu \left(\frac{\phi_1 + i\phi_2}{\sqrt{2}}\right)\right)$$

$$- m^2 \left(\left(\frac{\phi_1 - i\phi_2}{\sqrt{2}}\right)\left(\frac{\phi_1 + i\phi_2}{\sqrt{2}}\right)\right)$$

$$= (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

$$= (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi .$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial_\mu \partial^\mu \phi , \quad \frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi^* , \quad \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^*$$

$$\Rightarrow (\partial^2 + m^2)\phi = 0$$

$$(\partial^2 + m^2)\phi^* = 0$$

$$\Leftrightarrow (\partial^2 + m^2)\left(\frac{\phi_1 + i\phi_2}{\sqrt{2}}\right) = 0$$

$$\Leftrightarrow (\partial^2 + m^2)\phi_1 = 0 , \ (\partial^2 + m^2)\phi_2 = 0 .$$

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e) $\phi = \dfrac{\phi_1 + i\phi_2}{\sqrt{2}}$

$$\Rightarrow \phi = \frac{1}{\sqrt{2}} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \Big[a_{\vec{p}} e^{-ipx} + a_{\vec{p}}^{+} e^{ipx} + i b_{\vec{p}} e^{-ipx} + i b_{\vec{p}}^{+} e^{ipx} \Big]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[\left(\frac{a_{\vec{p}} + i b_{\vec{p}}}{\sqrt{2}} \right) e^{-ipx} + \left(\frac{a_{\vec{p}}^{+} + i b_{\vec{p}}^{+}}{\sqrt{2}} \right) e^{ipx} \right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(c_{\vec{p}}\, e^{-ipx} + d_{\vec{p}}^{+} e^{ipx} \right)$$

with $\quad c_{\vec{p}} = \dfrac{a_{\vec{p}} + i b_{\vec{p}}}{\sqrt{2}}, \quad d_{\vec{p}}^{+} = \dfrac{a_{\vec{p}}^{+} + i b_{\vec{p}}^{+}}{\sqrt{2}}$

$$\rightarrow c_{\vec{p}}^{+} = \frac{a_{\vec{p}}^{+} - i b_{\vec{p}}^{+}}{\sqrt{2}}, \quad d_{\vec{p}} = \frac{a_{\vec{p}} - i b_{\vec{p}}}{\sqrt{2}}$$

$$\Rightarrow [c_{\vec{p}}, c_{\vec{p}'}] = [d_{\vec{p}}, d_{\vec{p}'}] = 0$$

$$[c_{\vec{p}}, d_{\vec{p}}^{+}] = \frac{1}{2}\left([a_{\vec{p}}, a_{\vec{p}}^{+}] + (i)^2 [b_{\vec{p}}, b_{\vec{p}}^{+}] \right)$$
$$= \frac{1}{2}(2\pi)^3 (1 - 1)\, \delta(\vec{p} - \vec{p}') = 0.$$

analogously $[c_{\vec{p}}^{+}, d_{\vec{p}}] = 0,$

$$[c_{\vec{p}}, c_{\vec{p}'}^{+}] = \frac{1}{2}\left([a_{\vec{p}}, a_{\vec{p}'}^{+}] + [b_{\vec{p}}, b_{\vec{p}'}^{+}] \right)$$
$$= 2 \cdot \frac{1}{2} \cdot (2\pi)\, \delta^{(3)}(\vec{p} - \vec{p}') = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p})$$

anagolously

$$[d_{\vec{p}}, d_{\vec{p}'}^{+}] = (2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{p}')$$

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Problem 2

a) $\mathcal{L} = \frac{1}{2} Z^2 (\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2$.

$$\phi \longrightarrow \tilde{\phi} = \phi / Z$$

$$\mathcal{L} \longrightarrow \frac{1}{2} Z^2 (\partial_\mu \tilde{\phi})^2 - \frac{1}{2}\mu^2 \tilde{\phi}^2$$

$$\cancel{= \frac{1}{2} Z^2 (\partial_\mu (Z^{-1}\phi))}$$

$$= \frac{1}{2} Z^2 Z^{-2} (\partial_\mu \phi)^2 - \frac{1}{2}\left(\frac{\mu}{Z}\right)^2 \phi^2$$

$$= \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$ with

$$m = \frac{\mu}{Z}$$

Shouldn't $\phi = \tilde{\phi}/Z$?

This would give

$$\mathcal{L} = \frac{1}{2} Z^2 (\partial_\mu \phi)^2 - \frac{1}{2}\mu^2\phi^2$$

$$= \frac{1}{2} Z^2 (\partial_\mu \tilde{\phi})^2 \frac{1}{Z^2} - \frac{1}{2}\mu^2 \frac{\tilde{\phi}^2}{Z^2}$$

$$= \frac{1}{2} (\partial_\mu \tilde{\phi})^2 - \frac{1}{2} m \tilde{\phi}^2 \quad \left(\frac{\mu}{Z} =: m\right).$$

and then we would be talking about excitations of $\tilde{\phi}$.

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p^Z \left(a_{\vec{p}}^\dagger a_{\vec{p}}\right)$$

$$\omega_p^Z = \sqrt{\vec{p}^{\,2} + m^2} = \sqrt{\vec{p}^{\,2} + \left(\frac{\mu}{Z}\right)^2}$$ and

$$H|\vec{p}\rangle = \omega_p^Z |\vec{p}\rangle.$$

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b)

b1) $\mathcal{L} = \frac{1}{2} Z^2 (\partial_\mu \phi)^2 - \frac{1}{2} \mu^2 \phi^2$

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = Z^2 \partial_t \phi \Rightarrow \partial_t \phi = \frac{1}{Z^2} \pi$$

$$\mathcal{H} = \pi \partial_t \phi - \mathcal{L} = \frac{1}{2Z^2} \pi^2 + \frac{1}{2} Z^2 (\vec{\nabla} \phi)^2 + \frac{1}{2} \mu^2 \phi^2$$

$$\Rightarrow \underline{H = \frac{1}{2} \int d^3x \left(\frac{\pi^2}{Z^2} + Z^2 (\vec{\nabla} \phi)^2 + \mu^2 \phi^2 \right)}$$

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b2) $[\phi(\vec{x}'), H] = \frac{1}{2} \int d^3x \; \frac{1}{Z^2} [\phi, \pi^2] + Z^2 [\phi, (\vec{\nabla}\phi)^2] + \mu^2 [\phi, \phi^2]$

$$= \frac{1}{2} \int d^3x \; \frac{1}{Z^2} \left([\phi, \pi] \pi + \pi [\phi, \pi] \right) + 2 \cdot 0 + Z^2 [\phi, \vec{\nabla} \phi]$$

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$$= \frac{1}{2} \int d^3x \; \frac{1}{Z^2} \pi \cdot 2 \cdot i \cdot \delta^{(3)}(\vec{x} - \vec{x}') + 0$$

$$= i \; \pi(x') \frac{1}{Z^2}$$

$$= i \frac{Z^2}{Z^2} \partial_t \phi(x') = i \frac{d}{dt} \phi(x')$$

if and only if $[\phi(t, \vec{x}), \pi(t, \vec{x}')] = i \delta^{(3)}(\vec{x} - \vec{x}')$.

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$\cancel{[\phi, \vec{\nabla}\phi]} = \cancel{\ldots} \; f(\phi)$

$\cancel{= \hat{e}_i [\phi, \partial_i \phi]}$

$\boxed{\begin{aligned} * \; & \cancel{\vec{\nabla}[\phi, \phi] = \vec{\nabla}(\phi \cdot \phi) - \vec{\nabla}(\phi \cdot \phi)} \text{ see last page} \\ & \cancel{= (\vec{\nabla}\phi)\phi + \phi \vec{\nabla}\phi - (\vec{\nabla}\phi)\phi - \phi \vec{\nabla}\phi} \\ & \cancel{= [\nabla\phi, \phi] + [\phi, \nabla\phi] = 0 \rightarrow [\phi, \nabla\phi] = 0} \end{aligned}}$

Set $\phi(t,\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(b_{\vec p}\, e^{-ipx} + b^\dagger_{\vec p}\, e^{ipx} \right)$ with $\omega_p = \sqrt{p^2 + \frac{\mu^2}{Z^2}}$

and with $\quad [b_{\vec p}, b^\dagger_{\vec p'}] = (2\pi)^3 \delta^{(3)}(\vec p - \vec p') \cdot C$

as usual ~~(all other commutators vanish)~~

$$[b_{\vec p}, b_{\vec p'}] = [b^\dagger_{\vec p}, b^\dagger_{\vec p'}] = 0 .$$

we find

$$\pi(t,\vec x) = Z^2(-i)\int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{\omega_p}{2}}\left(b_{\vec p}\, e^{-ipx} - b^\dagger_{\vec p}\, e^{ipx}\right)$$

and

$$[\phi(t,\vec x), \pi(t,\vec y)]$$

$$= Z^2(-i)\frac{1}{2}\int \frac{d^3p\, d^3p'}{(2\pi)^6}\sqrt{\frac{\omega_{p'}}{\omega_p}}\left[b_{\vec p}\, e^{-ipx} + b^\dagger_{\vec p}\, e^{ipx},\ b_{\vec p'}\, e^{-ip'y} - b^\dagger_{\vec p'}\, e^{ip'y}\right]$$

$$= C Z^2(-i)\frac{1}{2}\int \frac{d^3p\, d^3p'}{(2\pi)^3}\sqrt{\frac{\omega_{p'}}{\omega_p}}\left[\left(-e^{-ipx}e^{ipy} - e^{ipx}e^{-ipy}\right)\delta(\vec p - \vec p')\right]$$

$$= Z^2 i \int \frac{d^3p}{(2\pi)^3}\left(e^{ip(x-y)} + e^{ip(y-x)}\right)\cdot \frac{1}{2} C$$

$$= Z^2 i\ \delta(\vec x - \vec y)\, C$$

$\rightarrow$ $C = 1/Z^2$ gives the canonical commutation relations. So set

$$[b_{\vec p}, b^\dagger_{\vec p'}] = 1/Z^2\, (2\pi)^3\, \delta^{(3)}(\vec p - \vec p') .$$

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$$H = \frac{1}{2}\int d^3x \; \frac{\Pi^2}{Z^2} + Z^2(\vec{\nabla}\phi)^2 + \mu^2\phi^2$$

$$\frac{1}{Z^2}\int d^3x \; \Pi^2 = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} (-i)^2 \sqrt{\frac{\omega_{\vec p}^2}{2}} \sqrt{\frac{\omega_{\vec p'}^2}{2}} \left(b_{\vec p}\, e^{-ipx} - b^\dagger_{\vec p}\, e^{ipx} \right)$$

$$\left(b_{\vec p'}\, e^{-ip'x} - b^\dagger_{\vec p'}\, e^{ip'x} \right) Z^4 / Z^2$$

$$= -\int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{\sqrt{\omega_{\vec p}^2 \omega_{\vec p'}^2}}{2} \Big[b_{\vec p} b_{\vec p'}\, e^{-ix(p+p')}$$

$$+ b^\dagger_{\vec p} b^\dagger_{\vec p'}\, e^{ix(p+p')}$$

$$- b^\dagger_{\vec p} b_{\vec p'}\, e^{ix(p-p')}$$

$$- b_{\vec p} b^\dagger_{\vec p'}\, e^{-ix(p-p')} \Big]$$

$$= -\int \frac{d^3p\, d^3p'}{(2\pi)^3} \frac{\sqrt{\omega_{\vec p}^2 \omega_{\vec p'}^2}}{2} \Big[\left(b_{\vec p} b_{-\vec p} + b^\dagger_{\vec p} b^\dagger_{-\vec p} \right) \delta^{(3)}(\vec p + \vec p')$$

$$- \left(b^\dagger_{\vec p} b_{\vec p} + b_{\vec p} b^\dagger_{\vec p} \right) \delta^{(3)}(\vec p - \vec p') \Big] Z^2$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\omega_p}{2} Z^2 \left[b^\dagger_{\vec p} b_{\vec p} + b_{\vec p} b^\dagger_{\vec p} - b_{\vec p} b_{-\vec p} - b^\dagger_{\vec p} b^\dagger_{-\vec p} \right]$$

$$Z^2 \int d^3x \, (\nabla\phi)^2 = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{i^2 \; \vec p \cdot \vec p'}{\sqrt{2\omega_{\vec p'}^2}\sqrt{2\omega_{\vec p}^2}} \left(b_{\vec p}\, e^{-ipx} - b^\dagger_{\vec p}\, e^{ipx} \right)$$

$$\left(b_{\vec p'}\, e^{-ip'x} - b^\dagger_{\vec p'}\, e^{ip'x} \right) Z^2$$

$$= + Z^2 \int \frac{d^3p}{(2\pi)^3} \frac{\vec p^{\,2}}{2\omega_{\vec p}} \Big[b_{\vec p} b_{-\vec p} + b^\dagger_{\vec p} b^\dagger_{-\vec p}$$

$$+ b^\dagger_{\vec p} b_{\vec p} + b_{\vec p} b^\dagger_{\vec p} \Big]$$

$$\mu^2 \int d^3x \; \phi^2 = \mu^2 \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{\omega_{p'}\omega_p}} \left(b_{\vec p}\, e^{-ipx} + b^\dagger_{\vec p}\, e^{ipx} \right)$$

$$\left(b_{\vec p'}\, e^{-ip'x} + b^\dagger_{\vec p'}\, e^{ip'x} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\mu^2}{2\omega_p} \left(b_{\vec p} b_{-\vec p} + b^\dagger_{\vec p} b^\dagger_{-\vec p} + b_{\vec p} b^\dagger_{\vec p} + b^\dagger_{\vec p} b_{\vec p} \right)$$

$$\Rightarrow \quad H = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \left(b_{\vec p} b_{-\vec p} + b^\dagger_{\vec p} b^\dagger_{-\vec p} \right) \Big[\overbrace{\mu^2 + Z^2 \vec p^{\,2} - Z^2 \omega_p^2}^{=0} \Big] \frac{1}{2\omega_p}$$

$$+ \left(b^\dagger_{\vec p} b_{\vec p} + b_{\vec p} b^\dagger_{\vec p} \right) \Big[\mu^2 + Z^2 \vec p^{\,2} + Z^2 \omega_p^2 \Big] \frac{1}{2\omega_p}$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left[b_{\vec p}^+ b_{\vec p} + b_{\vec p} b_{\vec p}^+\right]\left[2\omega_p^2 Z^2\right]\frac{1}{2\omega_p}$$

$$= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_p\left[b_{\vec p}^+ b_{\vec p} + b_{\vec p} b_{\vec p}^+\right]Z^2$$

$$= Z^2\int \frac{d^3p}{(2\pi)^3}\,\omega_p\left(b_{\vec p}^+ b_{\vec p} + \frac{1}{2}\left[b_{\vec p}, b_{\vec p}^+\right]\right)$$

$$\longrightarrow Z^2\int \frac{d^3\vec p}{(2\pi)^3}\,\omega_p\, b_{\vec p}^+ b_{\vec p}\,.$$

with $\omega_p = \sqrt{\vec p^{\,2} + \frac{\mu^2}{Z^2}}$

Then $[H, b_{p'}^+] =$

$$= Z^2\int \frac{d^3\vec p}{(2\pi)^3}\left[b_{\vec p}^+ b_{\vec p}, b_{\vec p'}^+\right]\omega_{\vec p}$$

$$= Z^2\int \frac{d^3\vec p}{(2\pi)^3}\, b_{\vec p}^+\,\omega_{\vec p}\,\delta^{(3)}(\vec p-\vec p\,')(2\pi)^3\frac{1}{Z^2}$$

$$= \omega_{\vec p'}\, b_{\vec p'}^+$$

$$\Rightarrow H|\vec p\rangle = H\, b_{\vec p}^+|0\rangle = [H, b_{\vec p}^+]|0\rangle$$

$$= \cancel{[b_{\vec p}^+, H]} = \omega_{\vec p}\, b_{\vec p}^+|0\rangle$$

$$= \omega_{\vec p}|\vec p\rangle\,,\ \text{so}$$

$\omega_{\vec p} = \sqrt{\vec p^{\,2} + \frac{\mu^2}{Z^2}}$ is the energy of the state $|\vec p\rangle$.

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$$\cancel{[\phi, \nabla\phi] = \phi(\nabla\phi) - (\nabla\phi)\phi}$$

$$\cancel{= \nabla(\phi\phi) - (\nabla\phi)\phi - (\nabla\phi)\phi}$$

$$\cancel{= \nabla(\phi\phi) - 2(\nabla\phi)\phi}$$

$$\cancel{= \nabla(\phi\phi) - (\nabla\phi)\phi - [\nabla\phi, \phi] - \phi(\nabla\phi)}$$

$$\cancel{= \nabla(\phi\phi) - \nabla(\phi\phi) + \phi(\nabla\phi) - (\nabla\phi)\phi}$$

$$\cancel{=}$$

$$\cancel{\phi(x), \nabla\phi(x')}$$

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \tilde{\phi}(\vec{p})\, e^{i\vec{p}\vec{x}}$$

$$\Rightarrow \nabla\phi(x) = \int \frac{d^3p}{(2\pi)^3} \tilde{\phi}(\vec{p})\, \nabla e^{i\vec{p}\vec{x}}$$

$$= \int \frac{d^3p}{(2\pi)^3}\, i\vec{p}\, \tilde{\phi}(\vec{p})\, e^{i\vec{p}x}$$

$$[\phi(x), \nabla\phi(x')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \left[\tilde{\phi}(\vec{p}), \tilde{\phi}(\vec{p}')\right] i\vec{p}'\, e^{i\vec{p}\vec{x}} e^{i\vec{p}'\vec{x}'}$$

$$= 0$$