QFT - Sheet 3

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Problem 1

$$(\partial^2 + m^2)\,\phi(t,\vec{x}) = 0$$

$$\phi(0,\vec{x}) = \phi_0(\vec{x}) \quad \partial_t \phi(t,\vec{x})\big|_{t=0} = \dot{\phi}_0(\vec{x}).$$

1.) $$\Delta(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p}\left(e^{-ip_\mu(x-y)^\mu} - e^{ip_\mu(x-y)^\mu}\right)$$

8 $\partial_\mu \partial^\mu \Delta(x-y) =$

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p}\left((-ip_\mu)(-ip^\mu)\, e^{-ip_\nu(x-y)^\nu} - (ip_\mu)(ip^\mu)\, e^{ip_\nu(x-y)^\nu}\right)$$

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{-\omega_p^2 + \vec{p}^2}{2E_p}\left(e^{-ip_\mu(x-y)^\mu} - e^{ip_\mu(x-y)^\mu}\right)$$

$$\Rightarrow (\partial_\mu \partial^\mu + m^2)\,\Delta(x-y) =$$

$$\int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p}\big(-\omega_p^2 + \underbrace{\vec{p}^2 + m^2}_{=\omega_p^2}\big)(\dots)$$

$$= 0. \qquad (\partial_\mu \partial^\mu + m^2) =: K$$

~~$\Delta(x-y)$~~

$$\phi(t,\vec{x}) = i\int d^3y \left[\dot{\phi}_0(\vec{y})\,\Delta(x-y) + \phi_0(\vec{y})\frac{\partial}{\partial t}\Delta(x-y)\right]$$

$$\Rightarrow K\phi(t,\vec{x}) = i\int d^3y \Big[\dot{\phi}_0(\vec{y})\, K\Delta(x-y) + \phi_0(\vec{y})\, \underbrace{K\frac{\partial}{\partial t}}_{=\frac{\partial}{\partial t}K}\Delta(x-y)\Big]$$

$$= i\int d^3y \left[\dot{\phi}_0(\vec{y}) \cdot 0 + \phi_0(\vec{y})\frac{\partial}{\partial t} \cdot 0\right]$$

$$= 0.$$

So $\phi(t,\vec{x})$ is a solution to the K-G-equation.

$$\phi(0,\vec{x}) = i\int d^3\vec{y}\,\Big[\dot{\phi}_0(\vec{y})\,\Delta(x-y)\Big|_{x_0=0} +$$
$$\phi_0(\vec{y})\Big[\frac{\partial}{\partial t}\Delta(x-y)\Big]\Big|_{x_0=0}\Big]$$

$$\Delta(x-y)\Big|_{x_0=0} = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\Big(e^{+ip_0y_0}\,e^{i\vec{p}(\vec{x}-\vec{y})} - e^{-ip_0y_0}\,e^{-i\vec{p}(\vec{x}-\vec{y})}\Big)$$
$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\,e^{i\vec{p}(\vec{x}-\vec{y})}\big(e^{ip_0y_0} - e^{-ip_0y_0}\big)$$

$$\frac{\partial}{\partial t}\Delta(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\Big((-ip_0)e^{-ip_\mu(x-y)^\mu} - (ip_0)\,e^{ip_\mu(x-y)^\mu}\Big)$$
$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\,ip_0\big(e^{-ip_\mu(x-y)^\mu} + e^{ip_\mu(x-y)^\mu}\big)$$

$$\frac{\partial}{\partial t}\Delta(x-y)\Big|_{x_0=0} = \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{2E_p}\Big(e^{ip_0y_0}\,e^{i\vec{p}(\vec{x}-\vec{y})} + e^{-ip_0y_0}\,e^{-i\vec{p}(\vec{x}-\vec{y})}\Big)$$
$$= \int \frac{d^3p}{(2\pi)^3}\frac{-ip_0}{2E_p}\,e^{i\vec{p}(\vec{x}-\vec{y})}\big(e^{ip_0y_0} + e^{-ip_0y_0}\big)$$

$$\Rightarrow \phi(0,\vec{x}) = i\int d^3\vec{y}\,\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\,e^{i\vec{p}(\vec{x}-\vec{y})}\times$$
$$\Big\{\big[\dot{\phi}_0(\vec{y}) - ip_0\,\phi_0(\vec{y})\big]e^{ip_0y_0} - \big[\dot{\phi}_0(\vec{y}) + ip_0\,\phi_0(\vec{y})\big]e^{-ip_0y_0}\Big\}$$

Assuming $y_0 = x_0 = t = 0$:

$$\phi(0,\vec{x}) = i\int d^3\vec{y}\,\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\,e^{i\vec{p}(\vec{x}-\vec{y})}\times$$
$$-2ip_0\,\phi_0(\vec{y})\,e^0$$
$$= i\int d^3\vec{y}\,\frac{d^3\vec{p}}{(2\pi)^3}(-i)\frac{\cancel{2p_0}}{\cancel{2E_p}}\,\phi_0(\vec{y})\,e^{i\vec{p}(\vec{x}-\vec{y})}$$
$$= \int d^3y\,\phi_0(\vec{y})\,\delta^{(3)}(\vec{x}-\vec{y})$$
$$= \phi_0(\vec{x})$$

$$\frac{\partial^2}{\partial t^2}\Delta(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}\left(-p_0^2\right)\left(e^{-ip_\mu(x-y)^\mu} - e^{ip_\mu(x-y)^\mu}\right)$$

$$\left.\frac{\partial^2}{\partial t^2}\Delta(x-y)\right|_{x_0=0} = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{-p_0}{2}\left(e^{ip_0y_0} - e^{-ip_0y_0}\right)e^{i\vec{p}(\vec{x}-\vec{y})}$$

$$\Rightarrow \left.\partial_t\, \phi(t,\vec{x})\right|_{t=0} =$$

$$i\int d^3\vec{y}\;\Big[\dot{\phi}_0(\vec{y})\left.\left(\frac{\partial}{\partial t}\Delta(x-y)\right)\right|_{x_0=y_0=0} + \phi_0(\vec{y})\left.\left(\frac{\partial^2}{\partial t^2}\Delta(x-y)\right)\right|_{x_0=y_0=0}\Big]$$

$$= i\int d^3\vec{y}\,\frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}(\vec{x}-\vec{y})} \times \left[\dot{\phi}_0(\vec{y})\,\frac{2(-ip_0)}{2E_p} + \phi_0(\vec{y})\cdot \overset{(0)}{\cancel{\frac{2(-p_0)}{2}}}\right]$$

$$= i\int d^3\vec{y}\,\frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}(\vec{x}-\vec{y})} \times \left[\dot{\phi}_0(\vec{y})(-i)\right] = \int d^3\vec{y}\,\frac{d^3\vec{p}}{(2\pi)^3}\,\dot{\phi}_0(\vec{y})\, e^{i\vec{p}(\vec{x}-\vec{y})}$$

$$= \int d^3\vec{y}\;\dot{\phi}_0(\vec{y})\cdot\delta^{(3)}(\vec{x}-\vec{y}) = \dot{\phi}_0(\vec{x}).$$

2. $\phi(t+\delta t, \vec{x})$

$$= i\int d^3\vec{y}\, \dot{\phi}_0(\vec{y})\Delta(x-y)\Big|_{x_0-y_0=\delta t} + \phi_0(\vec{y})\left[\frac{\partial}{\partial t}\Delta(x-y)\right]_{x_0-y_0=\delta t}$$

$$= i\int d^3\vec{y} \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \times$$

$$\left[\dot{\phi}_0(\vec{y})\cdot\left(e^{-ip_0\delta t}-e^{ip_0\delta t}\right) + \phi_0(\vec{y})\cdot\left(e^{-ip_0\delta t}+e^{ip_0\delta t}\right)(-ip_0)\right]$$

$$= i\int d^3\vec{y} \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \times$$

$$\dot{\phi}_0(\vec{y})\left(-ip_0\delta t - ip_0\delta t\right) + \phi_0(\vec{y})(-ip_0)(2)$$

$$= \int d^3y \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p} 2p_0\, e^{i\vec{p}(\vec{x}-\vec{y})} \times$$

$$\phi_0(\vec{y}) + \dot{\phi}_0(\vec{y})\,\delta t$$

$$= \phi_0(\vec{x}) + \dot{\phi}_0(\vec{x})\,\delta t$$

This means ϕ follows a local interaction principle. For infinitesimal times, only infinitesimal changes of ϕ are possible. This is equivalent to a ~~local~~ causality-preserving field, because no information about the rest of the wavefunction is needed to determine the immediate future of the wavefunction at point $\vec{x}$.

Problem 2

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 + j(x)\,\phi(x)$$

10 a) $\partial_\mu \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \partial^\mu \phi = \partial^2 \phi$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi + j(x)$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cancel{+} - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\Rightarrow \quad \partial^2 \phi + m^2 \phi - j(x) = 0$$

$$\Rightarrow \quad (\partial^2 + m^2)\,\phi(x) = j(x).$$

We choose the retarded GF:

$$D_R(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{\Theta(x^0-y^0)}{2E_p}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right)$$

$$\Rightarrow \quad \phi(x) = \phi_0(x) + i \int d^4y\, D_R(x-y)\, j(y).$$

20 $$= \phi_0(x) + i \int d^4y \frac{d^3\vec{p}}{(2\pi)^3} \frac{\Theta(x^0-y^0)}{2E_p}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right) j(y).$$

$$j(y) = j(t,\vec{y}) = \begin{cases} 0 & t < t_1,\ t > t_2 \\ j(\vec{y}) & t_1 < t < t_2 . \end{cases}$$

We want to construct ϕ after the acting of the source $\Rightarrow \Theta(x_0 - y_0) = 1$ over the whole integral.

$$\Rightarrow \quad \int d^4y\, e^{ipy} j(y) = \tilde{j}(p),$$

$$\int d^4y\, e^{-ipy} j(y) = \tilde{j}^*(p) \qquad (j(x) \text{ is real.})$$

$$\Rightarrow \phi(x) = \phi_0(x) + i \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_p}\left(e^{-ipx} \tilde{j}(p) - e^{ipx} \tilde{j}^*(p)\right)$$

$$j(y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipy}\, \tilde{j}(p)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\, \frac{dp_0}{2\pi}\, e^{-ip_0y_0} e^{i\vec{p}\vec{y}}\, \tilde{j}(p)$$

$$j(y) = j(\vec{y})\left(\theta(y_0 - t_1) - \theta(t_2 - y_0)\right)$$

$$\tilde{j}(p_0) = \int e^{ip_0y_0}\, \theta(y_0 - t_1)\, dy_0 - \int dy_0\, e^{ip_0y_0}\, \theta(t_2 - y_0)$$

$$\tilde{y}_0 = y_0 - t_1 \qquad\qquad \tilde{y}_0 = t_2 - y_0$$

$$= \int e^{ip_0\tilde{y}_0} e^{ip_0t_1}\, \theta(\tilde{y}_0)\, d\tilde{y}_0 - \int d\tilde{y}_0\, e^{-ip\tilde{y}_0}\, \theta(\tilde{y}_0)\, e^{ip_0t_2}$$

$$= e^{ip_0t_1} - e^{ip_0t_2} \int e^{ip_0\tilde{y}_0}\, d\tilde{y}_0\, \theta(\tilde{y}_0)$$

$$= \left(e^{ip_0t_1} - e^{ip_0t_2}\right)\left(\frac{1}{i\pi p_0} + \delta(p_0)\right)\pi$$

$$= \left(\pi\delta(p_0) + \frac{1}{ip_0}\right)\left(e^{ip_0t_1} - e^{ip_0t_2}\right)$$

$$\Rightarrow \quad \tilde{j}(p) = \tilde{j}(\vec{p})\left(e^{ip_0t_1} - e^{ip_0t_2}\right)\left(\pi\delta(p_0) + \frac{1}{ip_0}\right)$$

$$\Rightarrow \quad \phi(t,\vec{x})\Big|_{t>t_2} = \phi_0(t,\vec{x}) + i\int \frac{d^3\vec{p}}{(2\pi)^3}\, \frac{1}{2E_p}\left(e^{-ipx}\,\tilde{j}(p) - e^{ipx}\,\tilde{j}^*(p)\right)$$

c) $\phi(t,\vec{x}) = \phi_0(t,\vec{x}) + I(\tilde{j}(\vec{p}))$

$$\Rightarrow \phi_0(t,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}}\, e^{-ip_\mu x^\mu} + a^\dagger_{\vec{p}}\, e^{ip_\mu x^\mu} \right\}$$

$$\Rightarrow \phi(t,\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ e^{-ipx}\left(a_{\vec{p}} + \frac{i}{\sqrt{2\omega_p}} \tilde{j}(p)\right) + e^{ipx}\left(a^\dagger_{\vec{p}} - \frac{i}{\sqrt{2\omega_p}} \tilde{j}^*(p)\right) \right\}$$

Define $b_{\vec{p}} := a_{\vec{p}} + \frac{i}{\sqrt{2\omega_p}} \tilde{j}(p)$

$$\Rightarrow b^\dagger_{\vec{p}} = a^\dagger_{\vec{p}} - \frac{i}{\sqrt{2\omega_p}} \tilde{j}^*(p)$$

and $[b_{\vec{p}}, b^\dagger_{\vec{p}}] = [a_{\vec{p}}, a^\dagger_{\vec{p}}] = 1$

$$\Rightarrow H = \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_p \left(b^\dagger_{\vec{p}} b_{\vec{p}}\right)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \omega_p \left(a^\dagger_p - \frac{i}{\sqrt{2\omega_p}} \tilde{j}^*(p)\right)\left(a_p + \frac{i}{\sqrt{2\omega_p}} \tilde{j}(p)\right).$$

Because with $b_{\vec{p}}, b^\dagger_{\vec{p}}$ $\phi(t,\vec{x})$ and $\pi(t,\vec{x})$ are analogous to the free case.

d) Before $a_{\vec{p}}|0\rangle = 0$, now $A_{\vec{p}}|\tilde{0}\rangle = 0$.

$$A_{\vec{p}}|\tilde{0}\rangle = a_{\vec{p}} + \frac{i}{\sqrt{2\omega_p}} \tilde{j}(p)\, |\tilde{0}\rangle$$

$$= \frac{i}{\sqrt{2\omega_p}} \tilde{j}(p)|\tilde{0}\rangle + a_{\vec{p}}|\tilde{0}\rangle \stackrel{!}{=} 0$$

$$\Rightarrow a_{\vec{p}}|\tilde{0}\rangle = -\frac{i}{\sqrt{2\omega_p}} \tilde{j}(p)|\tilde{0}\rangle$$

$\Rightarrow |\tilde{0}\rangle$ is eigenstate of the annihilation operator $a_{\vec{p}}$ with eigenvalue $\frac{-i}{\sqrt{2\omega_p}} \tilde{j}(p)$.

These eigenstates are

$\Rightarrow \cancel{|\tilde{0}\rangle =} \sum_{i=1}^{n} \frac{\alpha_p^n a_p^{\dagger n}}{n!} |0\rangle = \exp(\alpha_p a_p^\dagger)|0\rangle$

with $\alpha_p = \frac{-i\tilde{j}(p)}{\sqrt{2\omega_p}}$ (coherent states).

All this holds for all annihilation operators ~~($a_p a_{p'}^\dagger$ ...)~~

$\Rightarrow |\tilde{0}\rangle = \cancel{\int \frac{d^3p}{(2\pi)^3} \exp(\alpha_p a_p^\dagger)|0\rangle}$

$= \exp \int \frac{d^3p}{(2\pi)^3} \alpha_p a_p^\dagger \;|0\rangle$

This is possible because $a_p a_{p'}^\dagger |0\rangle \propto \delta^{(3)}(\vec{p} - \vec{p}')$.

$$\Rightarrow |\tilde{0}\rangle = \exp\left[-i \int \frac{d^3p}{(2\pi)^3} \frac{\tilde{j}(p)}{\sqrt{2\omega_p}} a_p^\dagger\right] |0\rangle$$

in analogy to $|\tilde{0}\rangle = \prod_{i=1}^{n} \exp(a_i^\dagger \alpha_i)|0\rangle$

$= \exp\left(\sum_{i=1}^{n} a_i^\dagger \alpha_i\right)|0\rangle$

for the discrete case.

more details