Problem 1

a) In the interaction picture:

$$\phi^i(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a^i_{\vec{p}} e^{-ipx} + a^{i\dagger}_{\vec{p}} e^{ipx} \right)$$

with $\left[a^i_{\vec{p}}, a^j_{\vec{p}'}\right] = \left[a^{i\dagger}_{\vec{p}}, a^{j\dagger}_{\vec{p}'}\right] = 0$,

$$\left[a^i_{\vec{p}}, a^{j\dagger}_{\vec{p}'}\right] = (2\pi)^3 \delta^{ij} \delta^{(3)}(\vec{p} - \vec{p}')$$

This yields

$\langle 0 | T \phi^i(x) \phi^j(y) | 0 \rangle \propto \delta^{ij}$,

because we could commute the annihilation operators to the right if $i \neq j$.

On the other hand, if $i = j$:

$\langle 0 | T \phi^i(x) \phi^i(y) | 0 \rangle$ is obviously completely analogous to the Klein-Gordon-Field.

$$\langle 0 | T \phi^i(x) \phi^i(y) | 0 \rangle = \theta(x_0 - y_0) \langle 0 | \phi^i(x) \phi^i(y) | 0 \rangle + \theta(y_0 - x_0) \langle 0 | \phi^i(y) \phi^i(x) | 0 \rangle = D_F(x-y)$$

per definition. Also

$$\langle 0 | T \phi^i(x) \phi^j(y) | 0 \rangle = \langle 0 | N \phi^i(x) \phi^j(y) | 0 \rangle + \phi^i(x) \phi^j(y) \langle 0 | 0 \rangle = \phi^i(x) \phi^j(y)$$

which yields the given formula.

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b) $\langle \Omega | T\{\phi^i(x) \cdots \phi^j(y)\} | \Omega \rangle$

$$= \lim_{t \to \infty(1-i\epsilon)} \frac{\langle 0 | T\{\phi^i(x) \cdots \phi^j(y) \exp[-i\int_{-t}^{t} dt' \, H_I(t')]\} | 0 \rangle}{\langle 0 | T\{\exp[-i\int_{-t}^{t} dt' \, H_I(t')]\} | 0 \rangle}$$

for four fields expand the numerator in first order:

$$\langle 0 | T\{\phi^i(x_1)\phi^j(x_2)\phi^k(x_3)\phi^l(x_4)\left(1 - \frac{i\lambda}{4}\int d^4z \sum_{m,n} \phi^m(z)\phi^m(z)\phi^n(z)\phi^n(z)\right)\} | 0 \rangle$$

First term: $\langle 0 | T\{\phi^i(x_1)\phi^j(x_2)\phi^k(x_3)\phi^l(x_4)\} | 0 \rangle$

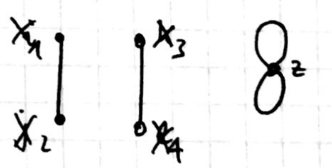
$$= \phi^i \phi^j \phi^k \phi^l + \phi^i \phi^j \phi^k \phi^l + \phi^i \phi^j \phi^k \phi^l$$

$$= \delta^{ij} D_F(x_1 - x_2)\delta^{kl} D_F(x_3 - x_4) + \delta^{ik}\delta^{jl} D_F(x_1 - x_3) D_F(x_2 - x_4)$$
$$+ \delta^{il}\delta^{jk} D_F(x_1 - x_4) D_F(x_2 - x_3).$$

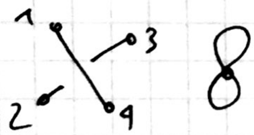
Second term:

$$\langle 0 | T\{(-\frac{i\lambda}{4})\int d^4z \sum_{m,n} \phi^i(x_1)\phi^j(x_2)\phi^k(x_3)\phi^l(x_4)\phi^m(z)\phi^m(z)\phi^n(z)\phi^n(z)\} | 0 \rangle$$

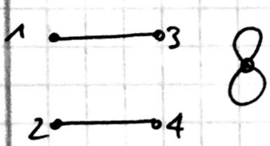
Possible contractions: ~~i with j, k with l~~,



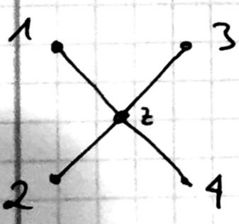
$\propto \delta^{ij}\delta^{kl}(1 + 2\delta^{mn})$



$\propto \delta^{il}\delta^{jk}(1 + 2\delta^{mn})$,



$\propto \delta^{ik}\delta^{jl}(1 + 2\delta^{mn})$,



$$\propto 4\delta^{im}\delta^{jm}\delta^{kn}\delta^{ln} + 4\delta^{im}\delta^{jn}\delta^{km}\delta^{ln}$$
$$+ 4\delta^{in}\delta^{jm}\delta^{km}\delta^{ln} + 4\delta^{im}\delta^{jn}\delta^{kn}\delta^{lm}$$
$$+ 4\delta^{in}\delta^{jm}\delta^{kn}\delta^{lm} + 4\delta^{in}\delta^{jn}\delta^{km}\delta^{lm}$$

we only consider the last term :

$$-\frac{i\lambda}{4}\int d^4z\; 4\sum_{m,n}\left(\delta^{\cdots}\right) D_F(x_1-z)\, D_F(x_2-z)\, D_F(x_3-z)\, D_F(x_4-z).$$

Switch to momentum space :

$$-i\lambda \sum_{m,n}\big(\delta^{im}\delta^{jm}\delta^{kn}\delta^{ln} + \delta^{in}\delta^{jn}\delta^{km}\delta^{lm} + \delta^{im}\delta^{jn}\delta^{km}\delta^{ln} + \delta^{in}\delta^{jm}\delta^{kn}\delta^{lm} + \delta^{im}\delta^{jn}\delta^{kn}\delta^{lm} + \delta^{in}\delta^{jm}\delta^{km}\delta^{ln}\big)$$

$$= -i\lambda\big(\delta^{ij}\delta^{kl} + \delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk} + \delta^{il}\delta^{jk}\big)$$

$$= -2i\lambda\big(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\big)$$

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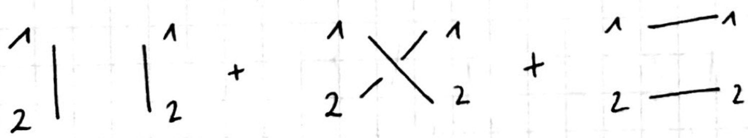
The other terms give a factor to what we calculated in 0th-order (but cancel with corresponding terms in the denominator, as they are all disconnected diagrams.).

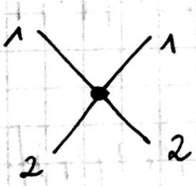
c) $\phi^1\phi^2 \longrightarrow \phi^1\phi^2$: $\langle\Omega|T\,\phi^1(x_1)\phi^2(x_2)\,\phi^1(x_3)\phi^2(x_4)|\Omega\rangle$

mom. space :



$\boxed{\text{zeroth order}}$

$= 0 + 0 + 1$



$= -2i\lambda$

$\boxed{\text{first order}}$

$$d\sigma = \frac{d\vec{p}_3 \, d\vec{p}_4}{(2\pi)^6 \, 4 E_3 E_4} (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) \times$$

$$\frac{|M(p_1 + p_2 \to p_3 + p_4)|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}$$

$$= \frac{d\vec{p}_3 \, d\vec{p}_4}{16\pi^2 E_3 E_4} \delta^{(4)}(p_3 + p_4 - p_1 - p_2) \times \frac{|M(p_1 + p_2 \to p_3 + p_4)|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}$$

c.o.m.: $p_1 = \left(\frac{E}{2}, \vec{0}, q\right), \quad p_2 = \left(\frac{E}{2}, \vec{0}, -q\right)$

$\Rightarrow \quad E = \sqrt{s} = p_1 + p_2 \qquad m_1 = m_2 = m$

$(p_1 \cdot p_2)^2 = \left(\frac{E^2}{4} + q^2\right)^2$

$m^2 = p_1^2 = \frac{E^2}{4} - q^2 \quad \Rightarrow \quad \frac{E^2}{4} + q^2 = m^2 + 2q^2$

$$\Rightarrow 4\sqrt{(p_1 \cdot p_2)^2 - m^4} = 4\sqrt{m^4 + 4q^4 + 4m^2q^2 - m^4}$$

$$= 4 \cdot 2\sqrt{q^2(m^2 + q^2)} = 4 \cdot 2q\sqrt{(m^2 + q^2)}$$

$$= 4 \cdot 2q\sqrt{\frac{E^2}{4}} = 4qE$$

$$\frac{d\vec{p}_3 \, d\vec{p}_4}{(2\pi)^6 \, 4 E_3 E_4} (2\pi)^4 \delta(E - E_3 - E_4) \delta^{(3)}(\vec{p}_3 + \vec{p}_4)$$

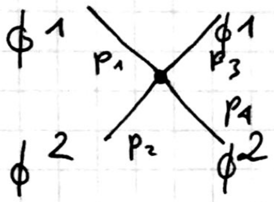
$$= \frac{d\vec{p}_3}{16\pi^2 E^2} \delta\left(E - \sqrt{\vec{p}_1^{\,2} + m^2} - \sqrt{\vec{p}_1^{\,2} + m^2}\right) = \frac{dp \, p^2 d\Omega}{16\pi^2 E^2} \delta(E - 2|p|)$$

$$= \frac{|\vec{p}_3| \, d\Omega}{16\pi^2 E} = \frac{\tilde{q} \, d\Omega}{16\pi^2 E}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\tilde{q}}{16\pi^2 E} \cdot \frac{1}{4qE} |M(p_1 + p_2 \to p_3 + p_4)|^2$$

$$\rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{\underset{64}{16}\pi^2 E^2} \frac{\tilde{q}}{q} |M(p_1 + p_2 \to p_3 + p_4)|^2$$

i) $iM(p_1+p_2 \to p_3+p_4)$ for $\phi^1\phi^2 \to \phi^1\phi^2$

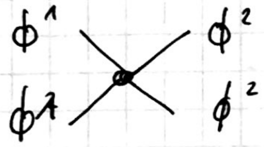
$$= \quad$$ 

$$= -2i\lambda$$

$$\Rightarrow \quad |M|^2 = 4\lambda^2$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{\lambda^2}{4\pi^2E^2}\frac{\tilde{q}}{q} = \frac{\lambda^2}{4\pi^2E^2} \quad \text{as } \tilde{q} = q$$

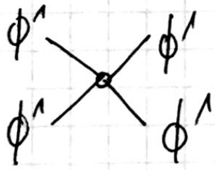
ii) $iM(\phi^1\phi^1 \to \phi^2\phi^2)$

$$= \quad$$ 

$$= -2i\lambda \rightarrow |M|^2 = 4\lambda^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2!}\frac{\lambda^2}{4\pi^2E^2}\frac{\tilde{q}}{q} = \frac{\lambda^2}{8\pi^2E^2}\frac{\tilde{q}}{q} = \frac{\lambda^2}{8\pi^2E^2}$$

iii) $iM(\phi^1\phi^1 \to \phi^1\phi^1)$

$$= \quad$$ 

$$= -2i\lambda(1+1+1) = -6i\lambda$$

$$\Rightarrow \quad |M|^2 = 36\lambda^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2!}\frac{36\lambda^2}{16\pi^2E^2}\frac{\tilde{q}}{q} = \frac{3\cdot3\cdot4}{2\cdot4\cdot4}\frac{\lambda^2}{\pi^2E^2}\frac{\tilde{q}}{q}$$

$$= \frac{9}{8}\frac{\lambda^2}{\pi^2E^2}$$

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Pr

Problem 2

$$V(\phi^2) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4}(\phi^2)^2$$

$$= -\frac{1}{2}\mu^2\sum_i \phi_i\phi_i + \frac{\lambda}{4}\sum_{i,j}\phi^i\phi^i\phi^j\phi^j$$

$$\frac{dV}{d\phi^N} = -\mu^2\phi^N + \frac{\lambda}{2}\sum_i \phi^i\phi^i\,\phi^N\cdot 2$$

$$+\lambda\phi^{N3}$$

$$= \phi^N\left(-\mu^2 + \lambda\phi^{N2} + \lambda\sum_i\phi^i\phi^i\right)$$

$= 0$ for $\phi^i = 0$ $\quad i = 1,\dots,N-1,$

$$\phi^N = \sqrt{\frac{\mu^2}{\lambda}} =: v$$

Expand:

$$V = -\frac{1}{2}\mu^2\sum_i\pi^i\pi^i - \frac{1}{2}\mu^2(v+\sigma)^2$$

$$+\frac{\lambda}{4}\sum_{i,j}\pi^i\pi^i\pi^j\pi^j + \frac{\lambda}{4}\sum_i\pi^i\pi^i(v+\sigma)^2\cdot 2$$

$$+\frac{\lambda}{4}(v+\sigma)^4$$

$$= \underline{-\frac{1}{2}\mu^2\sum_i\pi^i\pi^i} - \underline{\frac{1}{2}\mu^2v^2} - \underline{\frac{1}{2}\mu^2\sigma^2} - \underline{\mu^2 v\sigma}$$

$$+\frac{\lambda}{4}\sum_{i,j}\pi^i\pi^i\pi^j\pi^j + \underline{\frac{\lambda}{2}v^2\sum_i\pi^i\pi^i} + \frac{\lambda}{2}\sigma^2\sum_i\pi^i\pi^i$$

10 $$+\lambda v\sigma\sum_i\pi^i\pi^i + \underline{\frac{\lambda}{4}v^4} + \frac{\lambda}{4}\sigma^4 + \underline{\lambda v^3\sigma}$$

$$+\lambda v\sigma^3 + \underline{\frac{3\lambda}{2}v^2\sigma^2}$$

$$= 0 + 0 + \mu^2\sigma^2 - \frac{1}{4}\frac{\mu^4}{\lambda} + \frac{\lambda}{4}\sum_{i,j}\pi^i\pi^i\pi^j\pi^j$$

$$+\frac{\lambda}{2}\sigma^2\sum_i\pi^i\pi^i + \lambda v\sigma\sum_i\pi^i\pi^i + \frac{\lambda}{4}\sigma^4 + \lambda v\sigma^3$$

$$= \mu^2\sigma^2 + \lambda\left(\frac{1}{4}\sum_{i,j}\pi^i\pi^i\pi^j\pi^j + \frac{1}{2}\sum_i\pi^i\pi^i\sigma^2 + \frac{1}{4}\sigma^4\right)$$

$$+\sqrt{\lambda}\,\mu\left(\sum_i\pi^i\pi^i\sigma + \sigma^3\right) + \text{const.}$$

$\mu^2\sigma^2$ is the mass-term of the field σ,
all ~~other~~ interactions are cubic and quartic
and decrease with $\mathcal{O}(\sqrt{\lambda})$ or $\mathcal{O}(\lambda)$.

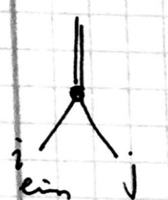
The mass of the field σ is
$m_\sigma = \sqrt{2}\,\mu$, $m_\pi = 0$

$$\langle 0|T\sigma(x)\sigma(y)|0\rangle = \overbracket{\sigma\,\sigma} = D_{F,\sigma}(x-y)$$

$$\xrightarrow{p} = \frac{i}{p^2 - m_\sigma^2 + i0} = \frac{i}{p^2 - 2\mu^2 + i0},$$

$$\langle 0|T\pi^i(x)\pi^j(y)|0\rangle = \overbracket{\pi^i\,\pi^j} = \delta^{ij} D_{F,\pi}(x-y)$$

$$i\xrightarrow{p}j = \frac{\delta^{ij}\, i}{p^2 + i0} \quad \text{as} \quad m_\pi = 0 \,.$$



$\hat{=}\ \langle \Omega | T\{\pi^i(x_1)\pi^j(x_2)\sigma(x_3)\} |0\rangle$

$$\underset{T\to\infty(1-i\varepsilon)}{\to} \frac{\langle 0|T\{\pi^i\pi^j\sigma \exp[-i\int_{-T}^{T} H_I(t)\,dt]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-T}^{T} H_I(t)\,dt]\}|0\rangle}$$

numerator in first order:

$$\langle 0|T\{\pi^i\pi^j\sigma \int d^4z\,(-i\sqrt{\lambda}\mu)(\sum_\ell \pi^\ell\pi^\ell\sigma + \sigma^3)\}|0\rangle$$

as only terms with an even number of fields can contribute. $\overbracket{\sigma\sigma}\overbracket{\sigma\sigma}\,\overbracket{\pi^i\pi^j}$ is a disconnected diagram,
so does not contribute either.

$$\Rightarrow \langle 0|T\{\pi^i\pi^j\sigma \int d^4z\,(-i\sqrt{\lambda}\mu)\sum_\ell \pi^\ell\pi^\ell\sigma\}|0\rangle$$

$$= -i\sqrt{\lambda}\mu \int d^4z\ \overbracket{\sigma\sigma}\Big(\sum_\ell \overbracket{\pi^i\pi^j}\,\overbracket{\pi^\ell\pi^\ell} + \sum_\ell \pi^i\pi^j\pi^\ell\pi^\ell$$

$$+ \sum_\ell \pi^i\pi^j\pi^\ell\pi^\ell\Big)$$

again $\pi^i\pi^j\,\pi^e\pi^e$ does not contribute because it is not connected.

$$\Rightarrow \quad -i\sqrt{\lambda}\mu \int d^4z \; 2\sigma\sigma \sum_e \delta^{ie}\delta^{je}\, D_{F,\pi}(x_1-z)\, D_{F,\pi}(x_2-z)$$

in momentum space:

$$-2i\sqrt{\lambda}\mu\; \delta^{ij} \quad \hat{=}$$ 

 $\hat{=} \; \langle 0 | T \{ \sigma(x_1)\sigma(x_2)\sigma(x_3) \exp[-i\int_{-T}^{T} H_I(t)\,dt] \} | 0 \rangle$

first order:

$$\langle 0 | T \{ \sigma\sigma\sigma \int d^4z\, (-i\sqrt{\lambda}\mu)\, \sigma^3 \} | 0 \rangle$$

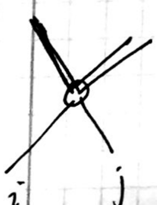
all external σ's must be contracted with an internal sigma. There are $3!$ possibilities for this.

$$\Rightarrow \quad -i\sqrt{\lambda}\mu\; 3! \int d^4z\; D_{F,\sigma}(x_1-z)\, D_{F,\sigma}(x_2-z)\, D_{F,\sigma}(x_3-z)$$

in mom space: $-6i\sqrt{\lambda}\mu$

same as in Problem 1:

$$-2i\lambda\,(\delta^{ij}\delta^{ke} + \delta^{ie}\delta^{jk} + \delta^{ik}\delta^{je})$$

 $\hat{=} \; \langle 0 | T \{ \sigma(x_1)\sigma(x_2)\, \pi^i(x_3)\pi^j(x_4) \exp[-i\int_{-T}^{T} H_I(t)\,dt] \} | 0 \rangle$

first order:

$$\langle 0 | T \{ \sigma\sigma\,\pi^i\pi^j \left(-i\frac{\lambda}{2}\right) \int d^4z \sum_e \pi^e\pi^e\sigma\sigma \} | 0 \rangle$$

$$\Rightarrow \quad -\frac{i\lambda}{2}\int d^4z \sum_e \left(\sigma\sigma\,\pi^i\pi^j\pi^e\pi^e\sigma\sigma \right)\cdot 2\cdot 2$$

where the factors are from the choice in which σ's one contracts and which π^i.

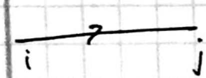
$\underline{\text{mom space} \Rightarrow}$ $\quad -\frac{i\lambda}{2} \sum_{e} \delta^{ie}\delta^{je} \cdot 4 = -2i\lambda\,\delta^{ij}$

$\underset{\text{1. ord.}}{\widehat{=}} \langle 0 | T \{ \sigma\sigma\sigma\sigma \left(-\frac{i\lambda}{4}\right) \int d^4z\, \sigma^4 \} | 0 \rangle$

$\overset{\text{mom. space}}{\Rightarrow} -\frac{i\lambda}{4} \cdot 4! = -6i\lambda$

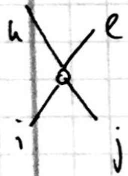
because one can choose 4 internal fields to contract the first external field with, then three and so on.

 $\widehat{=} \quad \frac{i}{p^2 - 2\mu^2 + i0}$,

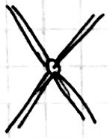


$\widehat{=} \quad \frac{\delta^{ij}\, i}{p^2 + i0}$,

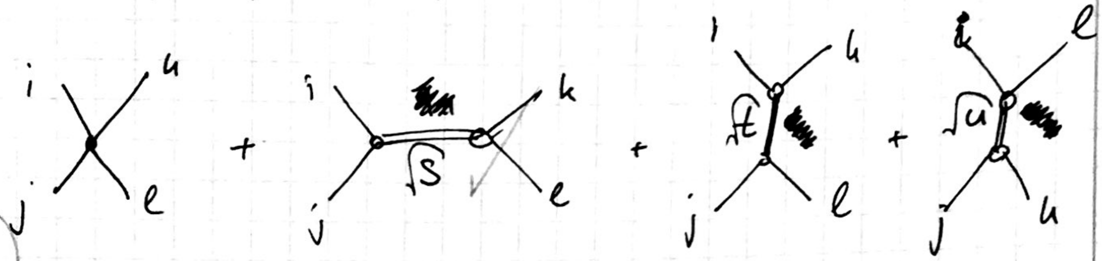
$= -2i\sqrt{\lambda}\mu\,\delta^{ij}$,  $= -6i\sqrt{\lambda}\mu$,



$= -2i\lambda\,(\delta^{ij}\delta^{hl} + \delta^{il}\delta^{jh} + \delta^{ih}\delta^{jl})$,

$= -2i\lambda\,\delta^{ij}$,  $= -6i\lambda$

c) $\pi^i(p_1)\,\pi^j(p_2) \to \pi^k(p_3)\,\pi^l(p_4) \;\hat{=}$



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d) $$iM = -2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}\right)$$
$$+\left(-4\lambda\mu^2\,\delta^{ij}\delta^{kl}\right)\cdot\frac{i}{s-2\mu^2+i0}$$
$$+\left(-4\lambda\mu^2\,\delta^{ik}\delta^{jl}\right)\cdot\frac{i}{t-2\mu^2+i0}$$
$$+\left(-4\lambda\mu^2\,\delta^{il}\delta^{jk}\right)\cdot\frac{i}{u-2\mu^2+i0}$$

$s=t=u=0$

$$= -2i\lambda\left(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{jk}\delta^{il}\right)$$
$$+\frac{4\mu^2 i\lambda}{2\mu^2}\left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)$$
$$= \left(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right)\left(-2i\lambda + 2i\lambda\right)$$
$$= 0$$

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~~$p_1 \to q_1 + q_2$~~

~~$p_1 = q_1 + q_2 = (E, \vec{0})$~~

~~$s = (p_1+p_2)^2 = E^2 - (\vec{p}_1 - \vec{p}_2)^2 = E^2$~~

At "threshold" $p_1 = p_2 = (0, \vec{0})$

$\Rightarrow s = t = u = 0^2 = 0$