Problem 1

a) $(i\gamma^\mu \partial_\mu - m)\psi = 0$

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$$\psi = \begin{pmatrix} \psi_L \\ -i\sigma_2 \psi_L^* \end{pmatrix} \qquad \partial_\mu = (\partial_t, \vec{\nabla}) \qquad \partial^\mu = (\partial_t, -\vec{\nabla})$$

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

define $\bar{\sigma}^\mu = (1, -\sigma^i)$,

$\sigma^\mu = (1, \sigma^i) \Rightarrow$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} i\sigma^\mu \partial_\mu (-i\sigma_2 \psi_L^*) - m\psi_L \\ i\bar{\sigma}^\mu \partial_\mu (\psi_L) - m(-i\sigma_2 \psi_L^*) \end{pmatrix} = 0$$

$$\Rightarrow \quad \sigma^\mu \partial_\mu \sigma_2 \psi_L^* - m\psi_L = 0,$$
$$\bar{\sigma}^\mu \partial_\mu \psi_L + m\sigma_2 \psi_L^* = 0$$ ✓

b) $\bar{\sigma}^\mu \partial_\mu e^{\pm ipx} = \bar{\sigma}^\mu e^{\pm ipx} \partial_\mu (p_\nu x^\nu)(\pm i)$

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$= \bar{\sigma}^\mu e^{\pm ipx} p_\nu \delta_\mu^\nu (\pm i)$

$= \pm i \bar{\sigma}^\mu p_\mu e^{\pm ipx}$

$= \pm i (E - \sigma^i \cdot (-\vec{p}_i)) e^{\pm ipx}$

$= \pm i (E + \vec{\sigma}\vec{p}) e^{\pm ipx}$ ✓ ok

c) $$\psi_L = \sum_{r=1,2} \int_p \left(a_r(p) \zeta_r(p) e^{-ipx} + a_r^\dagger(p) \eta_r(p) e^{+ipx} \right)$$

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$\bar\sigma^\mu \partial_\mu \psi_L =$

$$\sum_{r=1,2}\int_p \Big(-i(E+\vec\sigma\vec p)\, a_r(p)\, \zeta_r(p)\, e^{-ipx} + i(E+\vec\sigma\vec p)\, a_r^\dagger(p)\, \eta_r(p)\, e^{+ipx} \Big)$$

$$\psi_L^* = \sum_{r=1,2}\int_p \Big(a_r^\dagger(p)\, \zeta_r^*(p)\, e^{ipx} + a_r(p)\, \eta_r^*(p)\, e^{-ipx} \Big)$$

$$m\sigma_2\psi_L^* = \sum_{r=1,2}\int_p \Big(m\sigma_2\, a_r^\dagger(p)\, \zeta_r^*(p)\, e^{ipx} + m\sigma_2\, a_r(p)\, \eta_r^*(p)\, e^{-ipx} \Big)$$

$$\Rightarrow \quad -i(E+\vec\sigma\vec p)\,\zeta_r(p) + m\sigma_2\,\eta_r^*(p) = 0$$
$$i(E+\vec\sigma\vec p)\,\eta_r(p) + m\sigma_2\,\zeta_r^*(p) = 0$$

$$-i(E-\vec\sigma\vec p)(E+\vec\sigma\vec p)\,\eta_r(p) = (E-\vec\sigma\vec p)\, m\sigma_2\,\zeta_r^*(p)$$
$$-i(E^2-\vec\sigma^2\vec p^2)\,\eta_r(p) = (E-\vec\sigma\vec p)\, m\sigma_2\,\zeta_r^*(p)$$
$$-i\underbrace{(E^2-\vec p^2)}_{=m^2}\,\eta_r(p) = (E-\vec\sigma\vec p)\, m\sigma_2\,\zeta_r^*(p)$$
$$\Rightarrow \quad \eta_r(p) = \frac{(E-\vec\sigma\vec p)}{m}\, i\sigma_2\,\zeta_r^*(p)$$ ✓ good!

d) $\zeta_1 = \xi_-$, $\eta_2 = \xi_- \Rightarrow$

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$$\eta_1 = \left(\frac{E-\vec\sigma\vec p}{m}\right) i\sigma_2\,\xi_-^*$$

$$\xi_- = i\sigma_2\,\xi_+^* \Rightarrow \xi_-^* = -i\sigma_2^*\,\xi_+ \quad \Big|\; i\sigma_2\,\cdot$$

$$i\sigma_2\,\xi_-^* = -i\sigma_2\, i\sigma_2^*\,\xi_+ = \sigma_2\sigma_2^*\,\xi_+ = -\xi_+ \qquad \boxed{\sigma_2^* = -\sigma_2}$$

$$\Rightarrow \quad \xi_+ = -i\sigma_2\,\xi_-^*$$

$$\Rightarrow \eta_1 = -\left(\frac{E-\vec\sigma\vec p}{m}\right)\xi_+ = -\left(\frac{E-|\vec p|}{m}\right)\xi_+$$ ✓

$$\xi_- = \eta_2 = \frac{E-\vec\sigma\vec p}{m}\, i\sigma_2\,\zeta_2^*$$

$$\cancel{(E+\vec{\sigma}\vec{p})\xi_- = \frac{E^2-\vec{p}^2}{m} i\sigma_2 \xi_2^*}$$

analogous to c)

$$i(E+\vec{\sigma}\vec{p})\zeta_r(p) = m\sigma_2 \eta_r^*(p)$$

$$i(E^2-\vec{p}^2)\zeta_r(p) = (E-\vec{\sigma}\vec{p})\, m\sigma_2 \eta_r^*(p)$$

$$\zeta_r(p) = -i\frac{(E-\vec{\sigma}\vec{p})}{m}\sigma_2\eta_r^*(p)$$

$$\zeta_2(p) = -i\frac{E-\vec{\sigma}\vec{p}}{m}\sigma_2\eta_2^*(p)$$

$$= \frac{E-\vec{\sigma}\vec{p}}{m}\,-i\sigma_2\xi_-^*(p)$$

$$= \frac{E-\vec{\sigma}\vec{p}}{m}\xi_+(p)$$

$$= \frac{E-|\vec{p}|}{m}\xi_+ \quad \checkmark \text{ ok}$$

$$\zeta_1 = \xi_-,\quad \zeta_2 = \frac{E-|\vec{p}|}{m}\xi_+,\quad \eta_1 = -\frac{E-|\vec{p}|}{m}\xi_+$$

$$\eta_2 = \xi_-$$

$$\rightarrow \psi_L = \int_p \Big(a_1(p)\,\xi_-(p)\,e^{-ipx} - a_1^+(p)\left(\frac{E-|\vec{p}|}{m}\right)\xi_+ e^{+ipx}$$

$$+ a_2(p)\left(\frac{E-|\vec{p}|}{m}\right)\xi_+ e^{-ipx} + a_2^+(p)\,\xi_- e^{+ipx}\Big)$$

$$a_1 =: a_-,\quad a_2 =: a_+$$

$$\Rightarrow \psi_L = \int_p \Big(a_-\xi_- e^{-ipx} + a_+\xi_+\left(\frac{E-|\vec{p}|}{m}\right)e^{-ipx}$$

$$+ a_+^+\xi_- e^{+ipx} - a_-^+\left(\frac{E-|\vec{p}|}{m}\right)\xi_+ e^{+ipx}\Big)$$

$$\frac{E-|p|}{m} = \frac{(E-|\vec{p}|)(E+|\vec{p}|)}{m(E+|p|)} = \frac{E^2-\vec{p}^2}{m(E+|\vec{p}|)} = \frac{m}{E+|\vec{p}|}$$

$$\Rightarrow \psi_L = \int_p \Big(a_-\xi_- e^{-ipx} + a_+\xi_+\left(\frac{m}{E+|\vec{p}|}\right)e^{-ipx}$$

$$+ a_+^+\xi_- e^{ipx} - a_-^+\left(\frac{m}{E+|\vec{p}|}\right)\xi_+ e^{ipx}\Big)$$

correct! There was a typo in the ex!

✓ right! ☺

$$\psi_L = \int_p \zeta_- \left(a_- e^{-ipx} + a_+^\dagger e^{ipx} \right) + \zeta_+ \left(a_+ e^{-ipx} - a_-^\dagger e^{ipx} \right)$$