Theoretische Teilchenphysik I

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Exercise Sheet 1 | SS-2023 | Due date: 25.04.23

Phonons (100 Points)

Exercise 1.1: *(20 points)* A one-dimensional harmonic oscillator is described by the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2, \quad \dot{x} = \frac{d\,x}{d\,t} \tag{1}$$

(a) *(5 points)* Determine the canonical momentum p and the Hamiltonian H.

(b) *(10 points)* Creation and annihilation operators $a, a^\dagger$ are defined by the following equations

$$x = \sqrt{\frac{1}{2\omega}}\left(a + a^\dagger\right), \quad p = -i\sqrt{\frac{\omega}{2}}\left(a - a^\dagger\right). \tag{2}$$

Express the Hamiltonian H found in point (a) in terms of $a, a^\dagger$, satisfying commuting relation $\left[a, a^\dagger\right] = 1$. Work recursively to determine its eigenvalues and eigenvectors starting from a ground state $|0\rangle$ which satisfies the equation $a|0\rangle = 0$.

(c) *(5 points)* Show that $a(t) = ae^{-i\omega t}$ and $a^\dagger(t) = a^\dagger e^{i\omega t}$ provide solutions of time evolution equation for the operators a and $a^\dagger$

$$i\frac{d}{dt}a = [a, H], \quad i\frac{d}{dt}a^\dagger = [a^\dagger, H]. \tag{3}$$

Exercise 1.2: *(50 points)* Consider a system of coupled oscillators with the nearest-neighbors interactions (see Fig.1), described by the Lagrangian with the periodic boundary conditions

$$L = \frac{1}{2}\sum_{j=1}^{N}\dot{q}_j^2 - \frac{\nu^2}{2}\sum_{j=1}^{N}\left(q_{j+1} - q_j\right)^2, \quad q_{N+1}(t) = q_1(t), \quad \dot{q}_{N+1}(t) = \dot{q}_1(t). \tag{4}$$

The generalized coordinates q_j are the displacements of j-th mass, normalized to a square root of the mass m, from the stationary point $x_j = d \cdot j$.

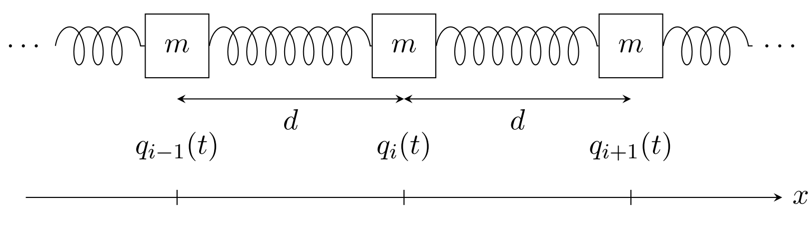

Figure 1: One-dimensional chain of springs

(a) *(5 points)* Use the Lagrangian in Eq. (4) to derive the classical equations of motion

$$\ddot{q}_j(t) = \nu^2\left[q_{j-1}(t) - q_j(t)\right] + \nu^2\left[q_{j+1}(t) - q_j(t)\right] \tag{5}$$

(b) *(5 points)* General solution of Eq. (5) can be written as a sum of particular solutions (referred to as "normal modes"). All particles in a normal mode solution oscillate with the same frequency. We label normal modes with a parameter $\alpha = 1, ..., N$ and write

$$q_j(t) = \sum_\alpha \mathrm{Re}\, q_j^\alpha(t) = \sum_\alpha \mathrm{Re}\left\{a_j^\alpha e^{i\omega_\alpha t}\right\}, \tag{6}$$

Use periodic boundary conditions (4) and equations of motion to show that eigenfrequencies ω^α and eigenamplitudes a_j^α of normal modes are described by the following equations

$$\omega_\alpha = 2\nu \left|\sin\frac{\alpha\pi}{N}\right|, \quad a_j^\alpha = C \cdot e^{-iKj}, \quad K = \frac{2\pi\alpha}{N}. \tag{7}$$

(c) *(5 points)* We can choose eigenamplitudes of normal modes to satisfy the orthogonality condition

$$\sum_{j=1}^{N} a_j^{\alpha*} a_j^\beta = \delta_{\alpha\beta}. \tag{8}$$

Use this condition to determine the constant C in Eq. (7).

(d) *(5 points)* Show that the completeness relation (7):

$$\sum_{\alpha=1}^{N} a_j^{\alpha*} a_k^\alpha = \delta_{jk}, \tag{9}$$

is satisfied by eigenamplitudes.

(e) *(20 points)* Introduce normal coordinates

$$Q_\alpha(t) = \sum_{j=1}^{N} a_j^\alpha q_j(t), \tag{10}$$

and show that the Lagrangian Eq. (4) can be written as follows

$$L = \frac{1}{2}\sum_{\alpha=1}^{N}\left(\dot{Q}_\alpha^* \dot{Q}_\alpha - \omega_\alpha^2 Q_\alpha^* Q_\alpha\right). \tag{11}$$

Note that Eq. (11) implies that normal coordinates are independent.

(f) *(10 points)* Use Eq. (11) to construct the conjugate momentum Π_α for the normal coordinate Q_α. We can quantize the system in Eq. (11) by requiring that Q_α, Π_β satisfy

$$[Q_\alpha, \Pi_\beta] = i\hbar\delta_{\alpha\beta}. \tag{12}$$

Use completeness relation (9) to invert Eq.(10) and express $\hat{q}_j$ and $\hat{p}_j$ in terms of Q_α and Π_α. Verify using Eq. (12) that $[\hat{q}_j, \hat{p}_k] = i\delta_{jk}$.

Exercise 1.3: *(30 points)* Finally, we discuss the *continuum limit* of the system describe by Eq. (4). To do this, we note that, as follows from Eq.(7), the phase difference between adjacent particles for a motion described by a normal mode α is $\frac{2\pi\alpha}{N}$. We can then define the wave length $\lambda = \frac{Nd}{\alpha}$ and the wave number $k = \frac{2\pi}{\lambda}$. We can now use k instead of α to label the modes. We write

$$a_{jk} = \frac{1}{\sqrt{N}} e^{-ikx_j}, \quad Q_k = \frac{1}{\sqrt{N}}\sum_{j=1}^{N} q_j e^{-ikx_j}, \quad q_j = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} Q_k e^{ikx_j}. \tag{13}$$

We define the *continuum limit* as $kd \ll 1$. In this limit, all displacements q_j can be described by a continuous function $\varphi(x)$:

$$q_j = q(x_j) \to \varphi(x), \quad \frac{q_{j+1} - q_j}{d} \to \frac{\partial\varphi}{\partial x}, \quad \sum_{j=1}^{N} \to \frac{1}{d}\int_0^L \mathrm{d}x \tag{14}$$

(a) *(15 points)* Introduce the density $\rho = 1/d$, and use Eq. ((14)) to show that the Lagrangian (4) can be written as

$$L = \int_0^L dx \left\{ \frac{\rho}{2} \left(\frac{\partial \varphi}{\partial t} \right)^2 - \frac{\rho c^2}{2} \left(\frac{\partial \varphi}{\partial x} \right)^2 \right\} \tag{15}$$

Give explicitly the parameter c in terms of the parameters of the original Lagrangian (4).

(b) *(15 points)* Derive the Euler-Lagrange equation for the field φ following from the Eq. (15) and give their general solutions. What is the role of the parameter c in Eq. (15)?