Theoretische Teilchenphysik I

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Exercise Sheet 2 | SS-2023 | Due date: 02.05.23

Casimir force (100 points)

Exercise 2.1: *(10 points)* In high-energy physics, one uses the so-called *HEP units*. For example, velocity is measured in units of the speed of light $c = 1$, and quantum effects in units of reduced Planck constant $\hbar \equiv \frac{h}{2\pi} = 1$. The basic unit that we employ when using the HEP units is electronvolts and its relatives (kiloelektrovolt (keV), megaelectrovolt (MeV) etc.). Numerically, $1\,\text{eV} = 1.602\,176 \times 10^{-19}\,\text{J}$.

(a) *(2 points)* Express one meter m and one second s through MeV. Note: $c = 299\,792\,458\,\text{m}\,\text{s}^{-1}$ and $\hbar = 6.582\,119 \times 10^{-22}\,\text{MeV}\,\text{s}$. What is the mass of 1 kilogram in HEP units?

(b) *(3 points)* The goal of the *KATRIN* experiment at KIT is to measure electron neutrino mass or set an upper bound. The minimal mass value which can be measured by *KATRIN* is $m_{\nu_e} = 0.35\,\text{eV}$. What is the measurable neutrino mass in kg?

(c) *(5 points)* Higgs bosons produced at the LHC typically move with the velocity $v \sim 0.6\,c$ where c is the speed of light. The Higgs boson width is $\Gamma_H = 4\,\text{MeV}$; it is related to its lifetime *in it rest frame* through $\Gamma_H = 1/\tau_H$. Estimate the mean distance (in m) that Higgs bosons travel before their decay in the laboratory frame.

Exercise 2.2: *(20 points)* The quantization of a free scalar field was discussed during the lectures. It was stated that, by writing the field operator as

$$\varphi(t,\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{\sqrt{(2\pi)^3}\sqrt{2\omega_p}} \left[a_p e^{-i(\omega_p t - \vec{p}\cdot\vec{x})} + a_p^\dagger e^{i(\omega_p t - \vec{p}\cdot\vec{x})}\right], \quad \omega_p = \sqrt{\vec{p}^2 + m^2}, \tag{1}$$

where creation and annihilation operators satisfy the commutation relations

$$\left[a_p, a_q^\dagger\right] = \delta^{(3)}(\vec{p} - \vec{q}), \quad [a_p, a_q] = 0, \tag{2}$$

one can re-write the Hamiltonian operator

$$H = \frac{1}{2}\int \mathrm{d}^3x \left[\pi^2(t,\vec{x}) + \left(\vec{\nabla}\varphi(t,\vec{x})\right)^2 + m^2\varphi^2(t,\vec{x})\right] \tag{3}$$

as

$$H = \int \mathrm{d}^3p\, \omega_p \left(a_p^\dagger a_p + \frac{1}{2}\delta^{(3)}(0)\right). \tag{4}$$

Perform all the steps of the derivation explicitely.

Exercise 2.3: *(30 points)* Consider a free massless scalar field in a $1+1$ dimensional theory, i.e. $x^\mu = (t, \vec{x})$. The theory is defined on an interval of length L. We assume that the field satisfies the following boundary conditions $\varphi(0,t) = \varphi(L,t) = 0$.

(a) *(5 points)* Use the monochromatic ansatz $\varphi_\omega(t,x) = e^{\pm i\omega t} f(x)$ and show that the solution of the wave equation for $\varphi(t,x)$ that satisfies the boundary conditions reads

$$\varphi_\omega(t,x) = N_{\omega_n}\, e^{\pm i\omega_n t} \sin\left(\frac{\pi n x}{L}\right), \tag{5}$$

where n is a positive integer, $\omega_n = \pi n/L$, and N_{ω_n} is a real normalisation factor that you can keep arbitrary at this point.

(b) *(10 points)* We employ the above solutions to construct an ansatz for the relativistic field

$$\varphi(t,x) = \sum_{n=1}^{\infty} N_{\omega_n}\left(\varphi_\omega\, a_n + \varphi_\omega^* a^\dagger\right) = \sum_{n=1}^{\infty} N_{\omega_n}\, \sin\left(\frac{\pi n x}{L}\right)\left[a_n e^{-i\omega_n t} + a_n^\dagger e^{i\omega_n t}\right], \quad (6)$$

where a_n and $a_n^\dagger$ are the creation and annihilation operators that satisfy the commutation relations

$$[a_n, a_m] = \left[a_n^\dagger, a_m^\dagger\right] = 0, \qquad \left[a_n, a_m^\dagger\right] = \delta_{nm}\,. \quad (7)$$

Use these commutation relations and the expression for $\varphi(t,x)$ in Eq. (6) to prove that

$$[\varphi(t,x), \pi(t,y)] = i\delta(x-y)\,, \quad (8)$$

for $0 < x, y < L$. You will need to use the so-called completeness relation for the sine functions

$$\sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{L}\right)\sin\left(\frac{\pi n y}{L}\right) = \frac{L}{2}\delta(x-y)\,, \quad (9)$$

and fix the normalisation factor N_{ω_n} appropriately. In particular, you should obtain $N_{\omega_n} = \sqrt{\frac{1}{\pi n}}$ in order to get the commutation relation in Eq. (8).

(c) *(5 points)* Show that the ansatz for the field φ and the commutation and annihilation relations for the creation operators diagonalize the Hamiltonian

$$H = \frac{1}{2}\int_0^L \mathrm{d}x\left(\pi^2(t,x) + (\partial_x\varphi(t,x))^2\right) = \sum_{n=1}^{\infty}\omega_n\left(a_n^\dagger a_n + \frac{1}{2}\right). \quad (10)$$

(d) *(10 points)* Use the above expression for the Hamiltonian to show that the vacuum energy in this case reads

$$E_{\text{vac}} = \langle 0|H|0\rangle = \sum_{n=1}^{\infty}\frac{\pi n}{2L}. \quad (11)$$

Can this sum be calculated?

Exercise 2.4: *(40 points)* In quantum field theory, the Casimir effect, also known as the Casimir force, is a physical force acting on the macroscopic boundaries of a confined space which arises from the quantum fluctuations of the field. The Casimir effect can be understood by the idea that the presence of macroscopic material interfaces, such as conducting metals, alters the vacuum expectation value of the energy of the quantized electromagnetic field. This effect manifests itself as a force between, for instance, two conducting plates in the vacuum. The Casimir effect can be studied in (3+1) dimensions by considering three plates in the set up shown in Fig. 1.

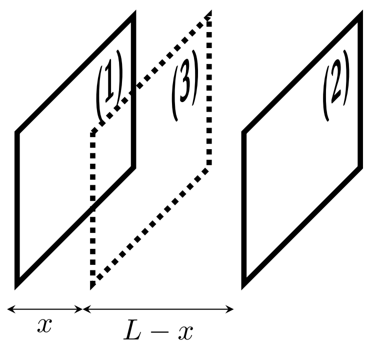

Figure 1: Sytem of three plates

Here for the sake of simplicity, we will only examine a (1+1)-dimensional problem.
Consider a (1+1)-dimensional massless scalar field theory. The space between two infinite plates in Fig. 1 confines the field into effective boxes of size x between plates $(1)-(3)$ and of size $L-x$ for space between plates $(2)-(3)$.

(a) *(10 points)* Using the result for the vacuum energy obtained in question (d) in the previous exercise, derive the expression for the energy of the plate (3) (Consider the system depicted in Fig.1 as two boxes of size x and $L-x$ respectively)

$$E = \sum_{n=1}^{\infty} \left[\frac{1}{2} \left(\frac{\pi n}{x} \right) + \frac{1}{2} \left(\frac{\pi n}{L-x} \right) \right] = f(x) + f(L-x)\,. \tag{12}$$

(b) *(10 points)* Regularize the infinite sum defining $f(x)$ by replacing $f(x) \to f_R(x,a)$ and introducing a smooth cutoff[1]

$$f_R(x,a) = \sum_{n=1}^{\infty} \frac{\pi n}{2x} e^{-\pi n \frac{a}{x}}\,. \tag{13}$$

Rewrite this sum as geometric series using differentiation with respect to the parameter a. Computing convergent geometric series, show that the expansion in the limit $a \to 0$ leads to

$$f_R(x,a) = \frac{x}{2\pi a^2} - \frac{\pi}{24x} + \mathcal{O}(a^2)\,. \tag{14}$$

(c) *(10 points)* From the regularized expression for the energy (12) derive the force between the two external plates in an approximation $L \gg x$

$$F = -\frac{\partial E}{\partial x} = -\frac{\pi}{24x^2}\,. \tag{15}$$

(d) *(10 points)* Restore proper units in the above expression and show that this force is a pure quantum effect. Estimate the magnitude of the force between the two external plates assuming $x = 0.1\mathrm{nm}$. Compare with the effect described in the paper "Casimir forces Still surprising after 60 years"

[1] Is known as a Heat-kernel regularization