Theoretische Teilchenphysik I

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Exercise Sheet 3 | SS-2023 | Due date: 09.05.23

Time ordered product (100 Points)

Exercise 3.1: *(35 points)* Using the equation of motion for scalar field and the definition of the T-product

$$T\{A(x)B(y)\} = \theta(x_0 - y_0)A(x)B(y) + \theta(y_0 - x_0)B(y)A(x)\,, \tag{1}$$

demonstrate that the following T-product of scalar fields is a Green's function of the Klein-Gordon equation

$$\left[\partial_\mu \partial^\mu + m^2\right] \langle 0 \,|\, T\phi(x)\phi(y) \,|\, 0\rangle = -i\delta(x-y)\,. \tag{2}$$

Note the following properties of the theta and delta functions:

$$\frac{d\theta(x-a)}{dx} = \delta(x-a), \quad \delta(x-y) = \delta(y-x), \quad f(x)\delta(x-a) = f(a)\delta(x-a). \tag{3}$$

Exercise 3.2: *(20 points)* In this exercise we want to show that the integration measure

$$\frac{d^3\vec{p}}{(2\pi)^3 2\omega_p}, \tag{4}$$

is invariant under Lorentz transformation. In particular, we want to prove this property for boosts in the z-axis direction.

(a) *(10 points)* Consider the momenta $\vec{p}$ and $\vec{p'}$ to be connected by Lorentz transform: $p'^\mu = \Lambda^\mu_\nu p^\nu$, where Λ is a boost in z-direction. Show that the following relation holds:

$$d^4p' = d^4p. \tag{5}$$

Hint: start with calculating $\det(\Lambda)$.

(b) *(10 points)* Use Eq.(5) to show explicitly that for boosts in the z-direction the folowing relation holds:

$$\frac{d^3\vec{p}}{(2\pi)^3 2\omega_p} = \frac{d^3\vec{p'}}{(2\pi)^3 2\omega_{p'}}\,. \tag{6}$$

Exercise 3.3: *(15 points)* Consider two points x and y, such that $x_0 > y_0$. In this case the time-ordered product from Eq.(1) reduces to a single term

$$\langle 0 \,|\, T\phi(x)\phi(y) \,|\, 0\rangle \to \langle 0 \,|\, \phi(x)\phi(y) \,|\, 0\rangle\;. \tag{7}$$

Consider now two new points $\tilde{x}$ and $\tilde{y}$, connected to original ones by Lorentz transform $\tilde{x}^\mu = \Lambda^\mu_\nu\, x^\nu$ and $\tilde{y}^\mu = \Lambda^\mu_\nu\, y^\nu$, and assume the transformation to be a boost along the z-axis. For $\tilde{x}, \tilde{y}$ show that the T-product will preserve the order of the fields. Take orthochronous transformation ($\Lambda^0_0 > 0$), from previous exercise, to show explicitly that $(\tilde{x}^0 - \tilde{y}^0) > 0$ and it follows that T-product will have the form:

$$\langle 0 \,|\, T\phi(\tilde{x})\phi(\tilde{y}) \,|\, 0\rangle \to \langle 0 \,|\, \phi(\tilde{x})\phi(\tilde{y}) \,|\, 0\rangle\;. \tag{8}$$

Exercise 3.4: *(30 points)* Repeat the derivation of the Feynman propagator considered in lectures, but now taking opposite sign of the imaginary shift

$$\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 - i\epsilon\, \mathrm{sgn}\,(p_0)} e^{-ip(x-y)}\,. \tag{9}$$