Theoretische Teilchenphysik I

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Exercise Sheet 5 | SS-2023 | Due date: 23.05.23

Non-relativistic limit of the Dirac equation (50 Points)

The goal of this exercise is to discuss solutions of the Dirac equation for the massive fermion in the (non-relativistic) limit when the fermion's three-momentum is much smaller than its mass , i.e. $|\vec{p}| \ll m$ or, equivalently, when its velocity is small compared to the speed of light.

In the non-relativistic limit, the energy of the fermion can be approximated by

$$p_0 = \sqrt{m^2 + \vec{p}^2} = m + \frac{\vec{p}^2}{2m} + \mathcal{O}\left(\frac{1}{m^2}\right) \tag{1}$$

To derive the non-relativistic limit of the Dirac equation it is convinient to factor out the mass dependent phase from the four component Dirac spinor using

$$\psi(t, \vec{x}) \equiv e^{-imt} \begin{pmatrix} \chi(t, \vec{x}) \\ \eta(t, \vec{x}) \end{pmatrix}, \tag{2}$$

Where now $\chi(x)$ and $\eta(x)$ are two-component spinors.

Exercise 5.1: *(10 points)* To study non-relativistic limit, it is useful to work with a different representation of gamma matrices, the so-called Dirac representation. The gamma matrices in this representation read

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{3}$$

Show that the above set of four matrices satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}. \tag{4}$$

Find a unitary matrix U such that

$$\gamma^\mu_W = U \gamma^\mu U^\dagger, \tag{5}$$

where γ^μ_W are the gamma matrices in the Weyl representaion (given in the script).
Hint: assume U to be real 4×4 block matrix with 2×2 blocks.

Exercise 5.2: *(20 points)* Consider the non-relativistic limit of the Dirac equation for the field $\psi(x)$. Start by deriving a system of equations for fields $\chi(x)$ and $\eta(x)$ defined in Eq. (2). Assuming that in the considered limit $i\partial\eta(t, \vec{x})/\partial t \ll m\eta$, eliminate the field $\eta(x)$ from the system of equations and show that the field $\chi(x)$ satisfies the following equation

$$i\frac{\partial}{\partial t}\chi(x) = \frac{1}{2m}\left(i\,\vec{\nabla} \cdot \vec{\sigma}\right)^2 \chi(x) \tag{6}$$

Exercise 5.3: *(20 points)* As we will learn later, to couple the Dirac fermion to the external electromagnetic field, we need to write

$$(i\gamma_\mu D^\mu + m)\Psi(x) = 0, \tag{7}$$

where

$$iD_t = i\partial_t - e\phi, \quad i\vec{D} = i\vec{\nabla} + e\vec{A}, \tag{8}$$

and ϕ and $\vec{A}$ are the electric potential and the vector potential.

Follow the previous exercise and show that in the non-relativistic limit the Dirac fermion coupled to the electormagnetic field satisfies the following (Pauli) equation

$$i\partial_t \chi(x) = \left[\frac{1}{2m}\left(i\,\vec{D}\right)^2 - g\frac{e}{2m}\frac{\vec{\sigma}\cdot\vec{B}}{2} + e\phi\right]\chi(x), \tag{9}$$

where we introduced the magnetic field $\vec{B} = \vec{\nabla} A$. Determine the value of the factor g.
Hint: use the following identity: $\sigma^k \sigma^n = \delta^{kn} + i \cdot \epsilon^{knl}\sigma^l$.

Majorana fermions (50 Points)

Exercise 5.4: *(20 points)* In the lecture, the Lorentz transformations of spinors were discussed. We have shown that two-componet spinors in the representations $(1/2,0)$ (ψ_L) and $(0,1/2)$ (ψ_R) transform differently.

(a) *(10 points)* Show, that if spinor ψ_L transforms according to $(1/2,0)$ representation rule, the spinor $\sigma_2\psi_L^*$ transforms according to $(0,1/2)$ representation rule.

Hence, the four-component Dirac spinor

$$\psi = \lambda \begin{pmatrix} \psi_L \\ i\sigma_2\psi_L^* \end{pmatrix} \tag{10}$$

is a valid spinor in $(1/2,1/2)$ representation. In the above equation λ is a constant factor to be determined later.

(b) *(10 points)* In the above ansatz for ψ, ψ_L is the only (two-component) complex spinor that can be adjusted to satisfy the Dirac equation whereas the Dirac equation mixes up all four components of a Dirac spinor. Show that by choosing ψ_L in the right way, ψ can satisfy the four-component Dirac equation.

Exercise 5.5: *(30 points)* As we already mentioned in the first excercise of this sheet, it is possible to use various representations of γ-matrices. Representations are related to each other by unitary transformations

$$\tilde{\gamma}^\mu = U\gamma^\mu U^\dagger \tag{11}$$

Consider a unitary transformation with the following matrix, where γ^μ are defined in Eq.(3)

$$U = \frac{1}{\sqrt{2}}\gamma_0\left(1+\gamma_2\right). \tag{12}$$

(a) *(10 points)* Compute matrices $\tilde{\gamma}^\mu$. Show that all these matrices are imaginary, i.e.

$$\tilde{\gamma}^{\mu,*} = -\tilde{\gamma}^\mu. \tag{13}$$

(b) *(10 points)* Consider the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0. \tag{14}$$

Using the matrix U, rewrite this equation as

$$\left(i\tilde{\gamma}^\mu\partial_\mu - m\right)\nu(x) = 0, \tag{15}$$

where $\nu(x) = U\psi(x)$. Show that the spinor $\nu(x)$ can be choosen to be real by a suitable choice of λ.

(c) *(10 points)* It follows from the above discussion that Majorana fermions can be described by real-valued spinors. Consider the Dirac equation for a Majorana fermion and couple it to the external electromagnetic field introduced in Eq.(8). Argue that the only consistent value for the electric charge e in this case is $e = 0$ so that a consistent theory of charged Majorana fermions does not exist.