Theoretische Teilchenphysik I

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Exercise Sheet 6 | SS-2023 | Due date: 06.06.23

Wrong quantisation of the Dirac field (30 Points)

Exercise 6.1: *(30 points)* As explained in the lectures, to quantise the Dirac field we use the anti-commutation relations for fields, canonical momenta etc. This is to be compared with the quantisation of a scalar field where commutation relations are used. The goal of this exercise is to study what happens if the Dirac field is quantized using the commutation relations.

(a) *(5 points)* Starting from the Lagrangian

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi\,, \tag{1}$$

derive the canonical momentum and write down the relevant equatl-time *commutation* relations.

(b) *(5 points)* Use the representation of the Dirac field in terms of creation and annihilation operators discussed in the lecture. Assuming that these operators satisfy standard commutation relations, i.e.

$$\left[a_{\vec{p},r}, a^{\dagger}_{\vec{q},s}\right] = \left[b_{\vec{p},r}, b^{\dagger}_{\vec{q},s}\right] = (2\pi)^3\delta^3(\vec{p}-\vec{q})\delta_{rs}\,, \tag{2}$$

and all other a and b operators commute, show that canonical commutation relations for fields and canonical momenta are satisfied.

(c) *(5 points)* The causality of theory requires that $\left[\psi(x), \bar{\psi}(y)\right]$ vanish for space-like intervals $(x-y)^2 < 0$. Verify that this is the case.

(d) *(5 points)* In the lecture, the Hamiltonian operator was quoted. Derive it showing all the steps explicitly.

(e) *(10 points)* Express the Hamiltonian in terms of creation and annihilation operators. Is there anything in this Hamiltonian that appears to be unacceptable?

Noether theorem and conserved charges (70 Points)

Exercise 6.2: *(20 points)* Noether theorem.

It was shown in the lecture that conserved energy momentum tensor

$$T^{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\varphi\right)}\partial^\nu\varphi - g^{\mu\nu}\mathcal{L}. \tag{3}$$

can be derived using Noether theorem.

(a) *(10 points)* Calculate the energy momentum tensor in the following theory

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu\varphi\right)^2 - \frac{m^2}{2}\varphi^2 + V(\varphi). \tag{4}$$

(b) *(10 points)* Conservation of the energy-momentum tensor implies

$$\partial_\mu T^{\nu\mu} = 0. \tag{5}$$

The above equation is a collection of *four* equations since we can choose the index ν freely. For each choice of the index ν write down the conserved "charge" associated with the corresponding Noether's current and explain its physical meaning.

Exercise 6.3: *(50 points)* Consider a theory described by the Lagrangian

$$\mathcal{L} = (\partial_\mu \varphi)^* (\partial_\mu \varphi) - m^2 \varphi^* \varphi, \tag{6}$$

where ϕ is the complex scalar field.

(a) *(10 points)* Perform canonical quantisation of the theory. In particular, show that writing the complex field in terms of two sets of creation and annihilation operators

$$\varphi(t, \vec{x}) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2\omega_p}} \left(a_p e^{-ip\cdot x} + b_p^\dagger e^{ip\cdot x} \right) , \tag{7}$$

one can satisfy canonical commutation relations and diagonalize the Hamiltonian.

(b) *(5 points)* The Lagrangian in Eq. (6) is invariant under phase redefinitions of the complex fields. Write the transformations that preserves the Lagrangian.

(c) *(5 points)* Use the Noether theorem to show that the conserved current related to the above symmetry reads

$$j_\mu(t, \vec{x}) = -i \left(\varphi \left(\partial_\mu \varphi \right)^* - \varphi^* \left(\partial_\mu \varphi \right) \right) . \tag{8}$$

(d) *(10 points)* Express the charge Q,

$$Q = \int d^3 x j_0(t, \vec{x}) \, , \tag{9}$$

in terms of creation and annihilation operators.

(e) *(10 points)* *Using canonical relations between fields and canonical momenta*, compute the commutation relation between Q and the Hamiltonian. Explain the meaning of the result.

(f) *(10 points)* Use the representation of the operator Q in terms of creation and annihilation operators to calculate

$$|X_1\rangle = Q\, a_p^\dagger \,|0\rangle \, , \quad |X_2\rangle = Q\, b_p^\dagger \,|0\rangle$$