Theoretische Teilchenphysik I

V: Prof. Kirill Melnikov, U: Dr. Andrey Pikelner, U: Dr. Chiara Signorile-Signorile

Exercise Sheet 8 SS-2023 Due date: 20.06.23

Feynman rules (60 points)

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{1}{2}M^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 - \frac{1}{2}g\phi_1^2\phi_2 \,, \tag{1}$$

that describes two real-valued fields interacting with each other.

(a) *(5 points)* Which term in the Lagrangian gives rise to an interaction? Describe the Hilbert space in a theory *without* the interacting term.

(b) *(5 points)* Use discussions in the lecture to guess the Feynman rules, including propagators and expressions for vertices that need to be used to construct Green's functions in a theory with the Lagrangian $\mathcal{L}$.

(c) *(10 points)* Consider the following Green's function in the interacting theory

$$\langle\Omega|\, T\hat{\phi}_1(x_1)\hat{\phi}_2(x_2)\hat{\phi}_1(x_3)\,|\Omega\rangle \,. \tag{2}$$

Use the Feynman rules that you have guessed and write down an expression for this Green's function through first order in the coupling constant g. Show the Feynman diagrams that you use in the calculation.

(d) *(10 points)* Use the representation of the Green's function $\langle\Omega|\, T\hat{\phi}_1(x_1)\hat{\phi}_2(x_2)\hat{\phi}_1(x_3)\,|\Omega\rangle$ in terms of fields in the interaction representation and the time evolution operator, to compute it to first order in g. Compare the result derived in the previous item to check the Feynman rules that you have guessed.

(e) *(15 points)* Consider the Green's function $\langle\Omega|\, T\hat{\phi}_1(x_1)\hat{\phi}_1(x_2)\,|\Omega\rangle$. Use Feynman rules that you guessed and provide the mathematical expression for this Green's function through second order in g. Draw all the Feynman diagrams and explain how the symmetry factor are computed (if needed).

(f) *(15 points)* Use the representation of the Green's function $\langle\Omega|\, T\hat{\phi}_1(x_1)\hat{\phi}_1(x_2)\,|\Omega\rangle$ in terms of fields in the interaction representation and the time evolution operator, to compute it to first order in g. Compare the result derived in the previous item to check the Feynman rules that you have guessed and the symmetry factors that you computed.

Linear Sigma Model with $m^2 > 0$ (40 points)

Consider a model of N real-valued scalar fields described by the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{N}(\partial_\mu \phi_i)(\partial^\mu \phi_i) - \frac{m^2}{2}\Phi^2 - \frac{\lambda}{4}(\Phi^2)^2, \tag{3}$$

where $\Phi^2 = \sum\limits_{i=1}^{N} \phi_i\phi_i$.

(a) *(10 points)* Describe the Hilbert space of this theory for $\lambda = 0$. Calculate

$$\langle\Omega|\, T\phi^i\phi^j\,|\Omega\rangle \tag{4}$$

to the zeroth order in λ.

(b) *(15 points)* Interaction vertices of the theory can be computed by considering Wick-contractions of the appropriate number of "external fields" with the Hamiltonian describing the interaction. Consider the following Green's function (the fields are considered to be fields in the interaction representation)

$$\langle 0|\, T(\phi^i(x_1)\phi^j(x_2)\phi^k(x_3)\phi^l(x_4)\, e^{-i\int d^4x \hat{H}_I(x)})\,|0\rangle\,. \tag{5}$$

Expanding this expression to first order in λ and selecting appropriate terms, verify that the interaction vertex in this theory is given by

$$-2i\lambda(\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl})\,. \tag{6}$$

(c) *(15 points)* Write an expression for the Green's function $\langle\Omega|\, T\phi^i\phi^j\,|\Omega\rangle$ through first order in λ. Make the dependence of your result on i, j and N explicit.