Theoretische Teilchenphysik I

V: Prof. Kirill Melnikov, U: Dr. Andrey Pikelner, U: Dr. Chiara Signorile-Signorile

Exercise Sheet 13 | SS-2023 | Due date: 25.07.23

Furry's theorem (40 points)

In this exercise, we study the interaction of vector and axial currents with a massive fermion. Consider the interaction of three vector currents. You can have a physical example by studying the light-by-light scattering $\gamma\gamma \to \gamma$, where γ represents a photon.

(a) *(5 points)* Can you find a term in the Standard Model Lagrangian describing the interaction of three photons at tree level?

(b) *(10 points)* Consider the scattering $\gamma\gamma \to \gamma$ at the first non-trivial order in the coupling constant. Draw the Feynman diagrams that contribute to the process and specify the Feynman rules that you need to compute the corresponding matrix element. Write down the matrix element for the process.

(c) *(10 points)* Argue why the process cannot occur. *Hint: focus on the Dirac trace that appears in the numerator of the matrix element.*

(d) *(5 points)* Consider the case where instead of photon you have three axial currents. Write the vertices you need to account for to compute the corresponding matrix element when mediated by a fermion loop.

(e) *(10 points)* Compute the matrix element under the assumptions of the previous questiona and verify whether it vanishes or not.

Light-by-light scattering (60 points)

We have seen in the previous exercise that interaction between three photons is forbidden in quantum electrodynamics. We will now check that the interaction between the *four* photons is actually possible. Consider the process

$$\gamma(k_1) + \gamma(k_2) \to \gamma(k_3) + \gamma(k_4). \tag{1}$$

(a) *(5 points)* Draw Feynman diagrams that contribute to this process (you should get three) and write mathematical expressions for them (as integrals over the loop momentum).

(b) *(20 points)* Choose any diagram and study its behaviour at large values of the loop momentum. You should find that the individual diagrams diverge. Use Pauli-Villars regularization and write the results for all diagrams in the following way

$$\text{Diagram} = e^4 \epsilon_{\mu_1}\, \epsilon_{\mu_2}\, \epsilon_{\mu_3}\, \epsilon_{\mu_4}\, r_{\mu_1\mu_2\mu_3\mu_4}\, \ln\frac{M^2}{\mu^2} + .., \tag{2}$$

where M is the mass of Pauli-Villars regulator, μ is an arbitrary quantity with mass dimension one and ellipses stand for M-independent terms. Compute tensor $r_{\mu_1\mu_2\mu_3\mu_4}$ explicitely. Use following replacement rules to calculate regularized integrals with loop momenta in numerator:

$$\int d^4l \frac{l^\mu l^\nu}{(l^2-m^2)^n} \to \frac{1}{4} g_{\mu\nu} \int d^4l \frac{l^2}{(l^2-m^2)^n} \tag{3}$$

$$\int d^4l \frac{l^\mu l^\nu l^\rho l^\sigma}{(l^2-m^2)^n} \to \frac{1}{24}\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right) \int d^4l \frac{l^4}{(l^2-m^2)^n}. \tag{4}$$

Hint:shift loop momenta if needed to bring integrals into the form (3) *and* (4).

(c) *(10 points)* Show that if you sum over all diagrams, the divergent contributions identified in the previous item cancel. Hence, we conclude that photon scattering on photons can be uniquely computed in QED.

(d) *(15 points)* This effect was one of the first computed after the advent of QED (Euler-Kolber-Heisenberg, 1935).

In case when the energy of photons is much smaller than the mass of the electron, the Euler-Heisenberg Lagrangian reads

$$\mathcal{L} = \frac{1}{8\pi}\left(\vec{E}^2 - \vec{B}^2\right) + \frac{\xi}{8\pi}\left((\vec{E}^2 - \vec{B}^2)^2 + 7(\vec{E}\vec{B})^2\right), \tag{5}$$

where $\xi = \alpha^2/(45\pi m^4)$ and $\vec{E}$ and $\vec{B}$ are "correct" electric and magnetic fields related to vector potentials

$$\vec{B} = \vec{\partial} \times \vec{A}, \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\partial} A_0. \tag{6}$$

Use this Lagrangian to *estimate* the photon-photon scattering cross section for visible light. In particular, show the dependence on the fine structure constant and on energies of incoming photons (assume that scattering occurs in the center of mass frame). Estimate a free path of a photon (visible light) that travels in a universe filled with cosmic microwave background radiation. Assume that microwave background photons have mean energy $E_\gamma \simeq 2.5\,\mathrm{eV}$.

(e) *(10 points)* In addition to the electron, assume the existence of a light charged fermion with mass $m_l = \kappa\, m_e, \kappa \ll 1$. Find the upper bound on the parameter κ from the requirement that the free path of the photon(now with additional interaction) is larger than the distance traveled by light in the lifetime of the universe.

BONUS exercise 1: Green's function (15 points)

The Green's function of the Klein-Gordon-equation can be written in the form

$$G(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{1}{k^2 - m^2}. \tag{7}$$

(a) *(5 points)* Performing the integration over k_0, we find poles of the integrand at $k_0 = \pm\sqrt{\vec{k}^2 + m^2}$. Specify how many integration paths around these poles are possible in the complex k_0-plane. Draw the pole configurations.

(b) *(10 points)* Show that the Green's function

$$G_D = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)} \frac{1}{k^2 - m^2 - i\epsilon} \tag{8}$$

can be expressed through the real (quantised) field φ and the vacuum state $|0\rangle$ as

$$iG_D(x-y) = -\Theta(x_0 - y_0)\langle 0|\varphi(y)\varphi(x)|0\rangle - \Theta(y_0 - x_0)\langle 0|\varphi(x)\varphi(y)|0\rangle\,. \tag{9}$$

BONUS exercise 2: scattering amplitudes and cross-sections (35 points)

Consider the process

$$\nu_\mu(k) + e^-(p) \to \nu_\mu(k') + e^-(p')\,, \tag{10}$$

where ν represent a neutrino, and e^- an electron. The process is mediated by a neutral current, that proceeds via the exchange of a neutral, massive Z boson, with mass m_Z. The corresponding Lagrangian reads

$$\mathcal{L}_{\mathsf{nc}} = -\frac{g}{4\cos\theta_W}\sum_{i=e,\mu}\bar{\nu}_i \not{Z}(1-\gamma^5)\nu_i + \bar{\ell}_i \not{Z}(4\sin^2\theta_W - 1 + \gamma^5)\ell_i \ , \tag{11}$$

with θ_W being the Weinberg's mixing angle, and g the coupling constant. The expression of the relevant interaction vertices are

$$Z\nu\nu: \qquad -i\frac{g}{4\cos\theta_W}(2\pi)^4\gamma^\mu(1-\gamma^5) \ , \tag{12}$$

$$Z\ell\ell: \qquad -i\frac{g}{4\cos\theta_W}(2\pi)^4\gamma^\mu(4\sin^2\theta_W - 1 + \gamma^5) \ . \tag{13}$$

(a) *(5 points)* Draw the Feynman diagram contributing to the process in Eq. (10) at the leading order in g, and specify the momentum flow. Write an explicit expression for the amplitude and its conjugated counterpart.

(b) *(5 points)* Compute the squared amplitude, summed over spin degree of freedom, and express it in terms of scalar products of momenta k, p, k', p'.

(c) *(5 points)* Define the Mandelstam invariants s, t, u and use them to express the scalar products of k, p, k', p'.

(d) *(10 points)* Compute the differential cross section $d\sigma/d\Omega$ and express it in terms of the Mandelstam invariants.

(e) *(10 points)* In the limit $s \ll m_Z^2$, assuming massless electrons, compute the total cross section and express it as a function of the Fermi constant G_F and the Weinberg angle.