Exercise Sheet 0

Exercises:
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Discussion: to be discussed

This zeroth exercise sheet is meant as a repetition of some topics that were discussed in the quantum mechanics courses. No points will be awarded for solving the exercises and you do not have to hand in your solutions, but you are encouraged to work on the exercise sheet as a warm up for the Theoretical Particle Physics I course.

Exercise 1: Klein paradox [no points]

We want to study an electron in an electrostatic step potential

$$V(\vec{r}) = q\,\Phi(\vec{r}) = V(z) = \begin{cases} V_0 & z \geq 0 \\ 0 & z < 0\,. \end{cases}$$

One stationary solution ($E \geq mc^2$) of the Dirac equation is of the form

$$\psi(\vec{r}) = e^{-iEt/\hbar}[\psi_i(z) + \psi_r(z) + \psi_t(z)].$$

Therein, we have the incoming, reflected and transmitted parts of the wave which propagates parallel to the z-axis, namely

$$\begin{aligned} \psi_i(z) &= a e^{ikz} r_0 \begin{pmatrix} 1 & 0 & \frac{\hbar ck}{E+mc^2} & 0 \end{pmatrix}^T && (z<0)\,, \\ \psi_r(z) &= e^{-ikz} \sum_{r=1}^{2} b_r\, w_r(-\hbar k) && (z<0)\,, \\ \psi_t(z) &= e^{ik_t z} \sum_{r=1}^{2} d_r\, w_r(\hbar k_t) && (z>0)\,. \end{aligned}$$

where we have $k > 0$, $k_t \in \mathbb{C}$,

$$r_0 = \sqrt{\frac{E+mc^2}{2mc^2}},$$

and the spinors

$$w_1(\hbar k) = r_0 \begin{pmatrix} 1 \\ 0 \\ \frac{c\hbar k}{E+mc^2} \\ 0 \end{pmatrix}, \qquad w_2(\hbar k) = r_0 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-c\hbar k}{E+mc^2} \end{pmatrix}.$$

The coefficients b_i and d_i $(i = 1, 2)$ can be determined by demanding continuity, i.e. $\psi_i + \psi_r = \psi_t$ at $z = 0$ and the result is[1]

$$b_1 = a\frac{1-R}{1+R}\,, \qquad d_1 = a\frac{2}{1+R}\sqrt{\frac{E+mc^2}{E-V_0+mc^2}}\,,$$
$$b_2 = 0, \qquad d_2 = 0,$$

where R is the reflection coefficient given by

$$R = \frac{k_t}{k}\frac{E+mc^2}{E-V_0+mc^2}\,.$$

(a) Using the Dirac basis, determine the current

$$\vec{j} = c\bar{\psi}(\vec{r},t)\,\vec{\gamma}\,\psi(\vec{r},t)$$

and separate it into the contributions $\vec{j}_i, \vec{j}_r, \vec{j}_t$, i.e. the incoming, reflected and transmitted part of the current. Express the componentwise ratios j_r/j_i and j_t/j_i in terms of R.

(b) Discuss the solutions of part (a) for the following situations: $V_0 < E - mc^2$, $E - mc^2 < V_0 < E + mc^2$, $V_0 > E + mc^2$.

(c) For the case $V_0 > E + mc^2$, show that the current is conserved at $z = 0$. Why would you consider the result as rather curious nonetheless (Klein paradox)?

Exercise 2: Dirac algebra in 4 dimensions **[no points]**

Independent of any representation, the Dirac matrices γ^μ and γ^5 in 4 space-time dimensions obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_4$ as well as $\{\gamma^5, \gamma^\mu\} = 0$. The γ^μ can always be chosen to be unitary so that $(\gamma^\mu)^\dagger = (\gamma^\mu)^{-1}$ holds.

(a) Prove the following Dirac algebra relations by using the unitarity and anti-commutation relations of the Dirac matrices, only:

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0\,, \qquad \gamma^5\gamma^5 = \mathbb{1}_4\,, \qquad \gamma_\mu\gamma^\mu = 4\cdot\mathbb{1}_4\,,$$
$$\gamma_\mu\gamma^\alpha\gamma^\mu = -2\gamma^\alpha\,, \qquad \gamma_\mu\gamma^\alpha\gamma^\beta\gamma^\mu = 4g^{\alpha\beta}\mathbb{1}_4\,.$$

(b) Prove the following trace identities by using the anti-commutation relations of the Dirac matrices and the general properties of a matrix trace, only:

$$\mathrm{Tr}\left(\gamma^\mu\right) = 0\,, \qquad \mathrm{Tr}\left(\gamma^5\right) = 0\,, \qquad \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^5\right) = 0\,, \qquad \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\right) = 0\,,$$
$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu}\,, \qquad \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)\,.$$

[1] Note that in order to arrive at this result, one has to use the replacement $E \to E - V_0$ for $z > 0$ in ψ_t.