Theoretical Particle Physics I

Prof. Dr. Milada Margarete Mühlleitner **WS 2023/24**

Exercise Sheet 1

Exercises:
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Please note that there will be ***no exercise class on Wednesday, November 1st****, due to the holiday. If you are interested in the discussion of sheet 0, please attend one of the exercise classes on Thursday, November 2nd, from 8:00–9:30 or 9:45–11:15.*

Exercise 1: Warm-up: collision of and decay into equal-mass particles [4P]

Two particles of the same mass m (e.g. particle and antiparticle) collide with opposite velocities of the same magnitude, $\mathbf{v}_1 = -\mathbf{v}_2$, with $|\mathbf{v}_1| = |\mathbf{v}_2| = v = \frac{3}{5}c$.

(a) **[2P]** What is the total energy $E = Mc^2$ of the combined system? Compare M with $2m$.

(b) **[2P]** Another particle at rest with mass M_0 decays into two particles of the same mass m. When is such a process possible? Calculate the velocity with which the two particles fly away from each other.

Exercise 2: Poincaré and Lorentz group [10P]

In the lecture, you have learned about the generators for the Lorentz and Poincaré group.

(a) **[5P]** Derive the commutators of the Poincaré algebra:

$$[P^\mu, M^{\rho\sigma}] = i\left(g^{\mu\rho}P^\sigma - g^{\mu\sigma}P^\rho\right), \tag{2.1}$$

$$[P^\mu, P^\nu] = 0\,, \tag{2.2}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} + g^{\mu\sigma}M^{\nu\rho}\right), \tag{2.3}$$

where P^μ denote the generators of translations and $M^{\mu\nu}$ the generators of Lorentz transformations. Do not use any specific representation of the P^μ or $M^{\mu\nu}$ (i.e. do not express them in terms of e.g. derivatives, Kronecker deltas, or metric tensors) to prove the relations.

Hint: Remind that a general Poincaré transformation (Λ, a) acts on a coordinate vector x^μ as $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, and that the multiplication rule for two Poincaré transformations is given as:

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2\Lambda_1, \Lambda_2 a_1 + a_2)\,.$$

Furthermore, note that for an infinitesimal transformation, you can write:

$$(1 + \delta\omega, \delta a) = 1 - \frac{i}{2}\delta\omega_{\alpha\beta}M^{\alpha\beta} + i\delta a_\mu P^\mu + \dots\,.$$

Try to use these expressions to relate the commutator of two generators to an expression that only contains one generator. Do not forget to antisymmetrise your expressions (i.e. sum over all permutations of antisymmetric indices to obtain the most general result)!

(b) **[2P]** By using the generators

$$L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\,,$$

determine the Lie algebra of the SO(3,1), i.e. prove that the $L_{\mu\nu}$ satisfy the same algebra as for the $M_{\mu\nu}$ above in Eq. (2.3),

$$[L_{\mu\nu}, L_{\rho\sigma}] = i\left(g_{\nu\rho}L_{\mu\sigma} - g_{\mu\rho}L_{\nu\sigma} - g_{\nu\sigma}L_{\mu\rho} + g_{\mu\sigma}L_{\nu\rho}\right)\,.$$

(c) **[3P]** The generators K^i and J^i of Lorentz boosts and rotations, respectively, can be expressed through the $M_{\mu\nu}$ and are given by:

$$K^i = M^{0i} \qquad \text{and} \qquad J^i = \frac{1}{2}\epsilon^{ijk}M_{jk} \qquad \text{for } i,j,k = 1,2,3\,,$$

with ϵ^{ijk} being the Levi-Civita tensor (with $\epsilon^{123} \equiv +1$). They satisfy the following algebra:

$$[J^i, K^j] = i\epsilon^{ijk}K^k\,, \qquad [K^i, K^j] = -i\epsilon^{ijk}J^k\,, \qquad [J^i, J^j] = i\epsilon^{ijk}J^k\,.$$

Out of the three commutators, prove two relations of your choice.

Exercise 3: Lorentz boosts **[6P]**

For this exercise assume that the boost operator K^2 is given by

$$K^2 = K_y = -i\left(t\frac{\partial}{\partial y} + y\frac{\partial}{\partial t}\right)\,.$$

(a) **[1P]** Calculate the effects of K_y and $(K_y)^2$ on the four-vector $x^\mu = (t,x,y,z)^T$.

(b) **[4P]** Determine the finite Lorentz transformation

$$x'^\mu = \exp\left(i\eta K_y\right)x^\mu\,,$$

where η is the *rapidity*, by using the results from part (a).

With the boost parameters $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, the boost can be written as

$$x'^\mu = \begin{pmatrix} \gamma & 0 & \gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ \gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x^\mu\,.$$

(c) **[1P]** Compare this alternative form of the boost with your result from (b). Show that the rapidity is given by

$$\eta = \frac{1}{2}\ln\frac{1+\beta}{1-\beta}\,.$$