Theoretical Particle Physics I

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Exercise Sheet 2

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Exercise 1: Gamma matrix representations [9P]

We consider the Dirac matrices γ^μ ($\mu = 0, ..., 3$) in the Weyl as well as the Dirac representation, respectively,

$$\gamma_\chi^\mu = \left(\begin{pmatrix} 0 & 1_{2\times2} \\ 1_{2\times2} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \right) , \qquad \gamma_D^\mu = \left(\begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \right) , \tag{1.1}$$

where $1_{2\times2}$ is the 2×2 unit matrix and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ with σ_i $(i = 1, 2, 3)$ represent the Pauli matrices.

(a) **[2P]** Show that both representations obey the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ by inserting the explicit representations separately into the anti-commutation relations.

(b) **[2P]** The Weyl and Dirac representations are connected by a unitary transformation U such that

$$\gamma_\chi^\mu = U^\dagger \gamma_D^\mu U$$

holds. Up to an arbitrary phase, determine the unitary 4×4 matrix U explicitly.

Hint: Remind that γ_χ^μ and γ_D^μ must have the same eigenvalues.

(c) **[1P]** A fifth Dirac matrix, γ^5, can be defined by $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that the explicit forms of γ_D^5 and γ_χ^5 for both the Dirac and the Weyl representation, respectively, are given by

$$\gamma_D^5 = \begin{pmatrix} 0 & 1_{2\times2} \\ 1_{2\times2} & 0 \end{pmatrix} , \qquad \gamma_\chi^5 = \begin{pmatrix} -1_{2\times2} & 0 \\ 0 & 1_{2\times2} \end{pmatrix} .$$

(d) **[2P]** We now define chirality projectors $\omega_\mp$ by

$$\omega_\mp := \frac{1_{4\times4} \mp \gamma^5}{2} ,$$

where ω_- is the left-chiral projector and ω_+ is the right-chiral projector. By using the properties of γ^5, show that the $\omega_\mp$ obey the following projector properties:

$$\omega_\mp^2 = \omega_\mp , \qquad \omega_-\omega_+ = \omega_+\omega_- = 0 .$$

(e) **[2P]** We now define a bispinor $\Psi := (\psi_L, \psi_R)$ in the Weyl basis, where ψ_L and ψ_R are left- and right-chiral spinors, respectively. Calculate the following bispinors by inserting the explicit representation of γ^5 in the Weyl basis:

$$\Psi_L := \omega_-\Psi \quad , \qquad \Psi_R := \omega_+\Psi .$$

Interpret the result and explain why the Weyl basis is often called chiral basis.

Exercise 2: Pauli-Lubanski operator **[11P]**

The *Pauli-Lubanski pseudovector* describes the spin state of a moving particle:

$$W_\mu = \frac{1}{2}\tilde{M}_{\mu\sigma}P^\sigma = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}M^{\nu\rho}P^\sigma,$$

where $M^{\mu\nu}$ and P^μ denote the generators of the Poincaré algebra, as introduced in the lecture. Its commutation relation is given as:

$$[W_\mu, W_\nu] = -i\varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma.$$

The simultaneous eigenvalues of P^2 and W^2 can be used to classify particles according to their mass and spin as irreducible representations of the Poincaré algebra.

We define the generalized Levi-Civita symbol in four dimensions as:

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} 1 \text{ if } \{\mu,\nu,\rho,\sigma\} \text{ is an odd permutation of } \{0,1,2,3\} \\ -1 \text{ if } \{\mu,\nu,\rho,\sigma\} \text{ is an even permutation of } \{0,1,2,3\} \\ 0 \text{ otherwise} \end{cases},$$

with $\varepsilon^{0123} = g^{\mu 0}g^{\nu 1}g^{\rho 2}g^{\sigma 3}\varepsilon_{\mu\nu\rho\sigma} = -\varepsilon_{0123}$.

(a) **[2P]** Show that the components of W_μ for a particle at rest are $(0, -m\vec{J})^T$, where $\vec{J} = \vec{x} \times \vec{P}$ is the total angular momentum operator in three dimensions.

(b) **[3P]** Prove the following identities:

(i) $W_\mu P^\mu = 0$,

(ii) $[W_\mu, P_\nu] = 0$.

(c) **[3P]** Show that P^2 and W^2 are the Casimir operators of the Poincaré group, *i.e.* that they commute with all its generators,

$$[P^2, P_\mu] = [P^2, M_{\mu\nu}] = 0 \qquad \text{and} \qquad [W^2, P_\mu] = [W^2, M_{\mu\nu}] = 0$$

You do not need to prove the last identity, $[W^2, M_{\mu\nu}] = 0$, as the calculation is really tedious.

(d) **[3P]** With

$$W^2 = -\frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 + M_{\mu\rho}M^{\nu\rho}P^\mu P_\nu,$$

show that

$$W^2|\vec{p} = \vec{0}, m, s\rangle = -m^2 s(s+1)|\vec{p} = \vec{0}, m, s\rangle,$$

where $|\vec{p} = \vec{0}, m, s\rangle$ is an eigenvector for a particle of mass m, spin s, and (vanishing) 3-momentum $\vec{p} = \vec{0}$, and $-m^2 s(s+1)$ is the corresponding eigenvalue.