Exercise Sheet 3

Exercises:
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Exercise 1: Poincaré and Lorentz group, continued [7P]

The generators $M_{\mu\nu}$ of Lorentz transformations can be split up into the three generators K^i of Lorentz boosts and the three generators J^i of rotations,

$$K^i = M^{0i} = -M_{0i} \qquad \text{and} \qquad J^i = \frac{1}{2}\,\epsilon^{ijk} M_{jk}\,,$$

where ϵ^{ijk} is the Levi-Civita tensor (with $\epsilon^{123} \equiv +1$). As shown in the lecture and on exercise sheet 1, these generators follow the commutation relations

$$\left[K^i, K^j\right] = -i\epsilon^{ijk} J^k\,, \quad \left[J^i, K^j\right] = i\epsilon^{ijk} K^k\,, \quad \left[J^i, J^j\right] = i\epsilon^{ijk} J^k\,.$$

Let us now define the operators N^a and $N^{a\dagger}$ with

$$N^a = \frac{1}{2}(J^a + iK^a) \quad \text{and} \quad N^{a\dagger} = \frac{1}{2}(J^a - iK^a) \quad \text{for } a = 1, 2, 3\,.$$

(a) **[2P]** Use the above relations to show the following commutation relations:

$$\left[N^i, N^{j\dagger}\right] = 0\,, \quad \left[N^i, N^j\right] = i\epsilon^{ijk} N^k\,, \quad \left[N^{i\dagger}, N^{j\dagger}\right] = i\epsilon^{ijk} N^{k\dagger}\,.$$

The operators N^i and $N^{i\dagger}$ thus both fulfill the Lie algebra of SO(3) ($\cong$ SU(2)).

(b) **[2P]** We now want to write a Lorentz transformation in terms of N^i and $N^{i\dagger}$. Show that:

$$-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu} = -i(\vec{\omega} - i\vec{\eta}) \cdot \vec{N} - i(\vec{\omega} + i\vec{\eta}) \cdot \vec{N}^\dagger,$$

where e.g. $\vec{\omega} \cdot \vec{N} = \omega_i N^i$ (with an implicit sum over i), and the components ω_i and η_i of the vectors $\vec{\omega}$ and $\vec{\eta}$, respectively, are defined through $\omega_{ij} = \epsilon_{ijk}\omega_k$ and $\omega_{0i} = \eta_i$.

The results of the previous exercises (a) and (b) allow us to write a Lorentz transformation as

$$U(\Lambda) = \exp\left(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right) = \exp\left(-i(\vec{\omega} - i\vec{\eta}) \cdot \vec{N}\right) \exp\left(-i(\vec{\omega} + i\vec{\eta}) \cdot \vec{N}^\dagger\right) =: U_L(\Lambda) U_R(\Lambda)$$

with the two factors

$$U_L(\Lambda) = \exp\left(-i(\vec{\omega} - i\vec{\eta}) \cdot \vec{N}\right) \quad \text{and} \quad U_R(\Lambda) = \exp\left(-i(\vec{\omega} + i\vec{\eta}) \cdot \vec{N}^\dagger\right).$$

Let us now discuss this transformation for fermions.

(c) **[3P]** A four-dimensional Dirac spinor Ψ is made up of two *Weyl* spinors ψ_L and ψ_R which transform separately according to $U_L(\Lambda)$ (or N^i) and $U_R(\Lambda)$ (or $N^{i\dagger}$), respectively. The general transformation rule for Ψ is given as

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \longmapsto \Psi' = \exp\left(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\Psi\,,$$

where the generator of Lorentz transformations $S^{\mu\nu}$ in a fermionic representation is given by $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.
Insert the Dirac matrices γ^μ in the Weyl representation (see e.g. exercise sheet 2) and show that the transformation decomposes as expected into two parts that act on the left- and right-handed Weyl spinors separately:

$$-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu} = \begin{pmatrix} -\frac{i}{2}(\vec{\omega} - i\vec{\eta})\cdot\vec{\sigma} & 0_{2\times 2} \\ 0_{2\times 2} & -\frac{i}{2}(\vec{\omega} + i\vec{\eta})\cdot\vec{\sigma} \end{pmatrix},$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$ denote the Pauli matrices.
Hint: The following relations may be helpful:

$$[\gamma^\mu, \gamma^\nu] = 2\gamma^\mu\gamma^\nu - 2g^{\mu\nu}\,, \qquad \sigma_i\sigma_j = \delta_{ij}\mathbb{1}_{2\times 2} + i\epsilon_{ijl}\sigma_l.$$

Note: For left-handed Weyl spinors, the generators N^i and $N^{i\dagger}$ can be represented by $N^i = \frac{\sigma_i}{2}$ and $N^{i\dagger} = 0$. For right-handed Weyl spinors, the generators are $N^i = 0$ and $N^{i\dagger} = \frac{\sigma_i}{2}$.
Note 2: The choice of a 4-dimensional representation for the Dirac spinor is not related to Minkowski space!

Exercise 2: Lagrangian of a massive vector field **[7P]**

The Lagrangian of a massive free vector field $V^\mu(x)$ is given by

$$\mathcal{L}_V = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m_V^2}{2}V_\mu V^\mu$$

where $m_V \neq 0$ denotes the mass of the vector particle and $F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu$ denotes the field-strength tensor.

(a) **[2P]** Calculate the equations of motion for V^μ, the so-called Proca equations.

(b) **[2P]** Using the equations of motion, prove that

$$\partial_\mu V^\mu = 0\,.$$

Using this condition, show that V^μ satisfies the Klein-Gordon equation.

A new Lagrangian $\mathcal{L} = \mathcal{L}_V + \mathcal{L}_D$ is given by adding a Dirac term

$$\mathcal{L}_D = \bar{\psi}\left(i\partial\!\!\!/ - m_D\right)\psi - q\bar{\psi}\gamma^\mu\psi V_\mu\,,$$

with an additional coupling term with a constant q between the spinor ψ and the vector field V_μ.

(c) **[1P]** Consider ψ, $\bar{\psi}$ and V_μ as independent fields and calculate the new equations of motion for these three fields.

(d) **[2P]** The vector current j^μ and axial vector current $j^{5\mu}$ can be defined as

$$j^\mu = \bar{\psi}\gamma^\mu\psi \qquad j^{5\mu} = \bar{\psi}\gamma^\mu\gamma_5\psi.$$

By using the equations of motion, prove that j^μ is a conserved quantity, whereas $j^{5\mu}$ is not conserved in general. In which special case is $j^{5\mu}$ conserved, as well?

Exercise 3: The σ model **[6P]**

The so-called "σ model" is made up of a massless Dirac fermion field ψ which couples to a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\sigma + i\pi)$. Here, σ and π are two real scalar fields denoting the real and imaginary parts of ϕ, respectively. The model can be described by the following Lagrangian:

$$\mathcal{L} = \mathcal{L}_\psi + \mathcal{L}_\phi + \mathcal{L}_I,$$

where

$$\mathcal{L}_\psi = \bar{\psi}i\partial\!\!\!/\psi, \qquad \mathcal{L}_\phi = (\partial_\mu\phi)^\dagger(\partial^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2,$$

with real parameters μ and λ. The interaction term between ψ and ϕ is given as:

$$\mathcal{L}_I = g(\bar{\psi}\psi\sigma + i\bar{\psi}\gamma_5\psi\pi), \tag{3.1}$$

where g is a real dimensionless coupling constant, and γ_5 is the special Dirac matrix which is defined as e.g. on exercise sheet 0. We want to look at the following transformations of the fields ϕ,

$$\phi \longrightarrow \phi' = e^{i\theta}\phi, \qquad \phi^\dagger \longrightarrow \phi^{\dagger\prime} = e^{-i\theta}\phi^\dagger\,, \tag{3.2}$$

and ψ,

$$\psi \longrightarrow \psi' = e^{i\beta\gamma^5}\psi\,. \tag{3.3}$$

(a) **[1P]** How do the real fields σ and π transform under the transformation of Eq. (3.2)?

(b) **[1P]** What does the transformation of Eq. (3.3) look like for the adjoint field $\bar{\psi}$?

(c) **[2P]** The interaction term $\mathcal{L}_I$ is not invariant under the *independent* transformations (3.3) and (3.2). The invariance can however be restored by a *simultaneous* transformation of the fields ψ, $\bar{\psi}$ and ϕ, $\phi^\dagger$. Find the linear relationship between the transformation parameters β and θ which leaves $\mathcal{L}$ invariant.

Hint: Consider infinitesimal transformations, i.e. neglect terms of order $\mathcal{O}(\beta^2, \theta^2, \beta\theta)$ or higher.

(d) **[2P]** The relation between β und θ defines a symmetry of $\mathcal{L}$. Determine the corresponding Noether current.