Theoretical Particle Physics I

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Exercise Sheet 4

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Exercise 1: Dilatation involving a real scalar field [5P]

We consider the Lagrangian of a real scalar field given by

$$\mathcal{L}_S = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m_S^2 \, \phi^2 - \frac{1}{4}\lambda \phi^4 \,.$$

(a) **[1P]** Derive the equation of motion for ϕ and the energy-momentum tensor $T^{\mu\nu}$ defined through

$$T^{\mu\nu} = \frac{\partial \mathcal{L}_S}{\partial(\partial_\mu \phi)}(\partial^\nu \phi) - g^{\mu\nu}\mathcal{L}_S \,.$$

(b) **[2P]** Show that the action $S = \int \mathrm{d}^4 x \, \mathcal{L}_S$ is invariant under dilatations for $m_S = 0$, i.e. under the transformations

$$x'_\mu = e^{-\alpha} x_\mu \,, \qquad \phi'(x') = e^{\alpha}\phi(x) \,.$$

(c) **[1P]** Show that for $m_S = 0$ the Noether current for the dilatation given in the previous subexercise is given by

$$j^\mu = T^{\mu\alpha} x_\alpha + \frac{1}{2}\partial^\mu(\phi^2) \,.$$

(d) **[1P]** Use the energy-momentum tensor and the equation of motion to show that in the massive case only the mass term breaks the invariance under dilatations, i.e. $\partial_\mu j^\mu = m_S^2 \phi^2$.

Exercise 2: Assembly instructions for a Lagrangian [7P]

The Lagrangian density $\mathcal{L}$ has several properties, some of which are:

- $\mathcal{L}$ must transform as a scalar under Lorentz transformations, and it must be invariant under possible internal symmetries.
- $\mathcal{L}$ must be of mass dimension 4. Note that we will use natural units ($\hbar = c = 1$) from now on, i.e. when we say that something is *dimensionless* or *of dimension* x, we always refer to the dimension of mass (or equivalently energy).
- All fields appearing in the terms of $\mathcal{L}$ have to be evaluated at the same space-time point.

(a) **[1P]** For a Dirac spinor $\psi(x)$, a Lorentz transformation is given as

$$\psi(x) \to \psi'(x') = S(\Lambda)\psi(x)\,,$$

where $S(\Lambda)$ is a 4×4 matrix in spinor space which depends on the Lorentz transformation Λ (see also e.g. sheet 3, exercise 1). We are only interested in proper orthochronous Lorentz transformations $S_L \equiv S(\Lambda_L)$ and parity transformations $S_P \equiv S(\Lambda_P)$ with $\Lambda_P = \mathrm{diag}(1,-1,-1,-1)$. The following relations are given:

$$\begin{aligned} S^{-1}(\Lambda)\gamma^\mu S(\Lambda) &= \Lambda^\mu{}_\nu\gamma^\nu && \text{for general } S\,, \\ S^{-1} &= \gamma^0 S^\dagger\gamma^0 && \text{both for } S_L \text{ and } S_P\,, \\ [S_L,\gamma^5] &= 0 && \text{only for } S_L\,, \\ \{S_P,\gamma^5\} &= 0 && \text{only for } S_P\,. \end{aligned} \tag{2.1}$$

For parity transformations S_P, derive the following commutator and anticommutator relations for the temporal and spatial components of the γ matrices:

$$[S_P,\gamma^0] = 0\,, \qquad \{S_P,\gamma^k\} = 0 \quad \text{for } k=1,2,3\,.$$

(b) **[3P]** Show that, using the properties of part **(a)**, the following bilinear forms transform as denoted under a Lorentz transformation:

(i) $\bar\psi\psi$: scalar,

(ii) $\bar\psi\gamma^5\psi$: pseudoscalar,

(iii) $\bar\psi\gamma^\mu\psi$: vector,

(iv) $\bar\psi\gamma^5\gamma^\mu\psi$: axial vector,

(v) $\bar\psi\sigma^{\mu\nu}\psi$: (antisymmetric) tensor of rank 2.

What about $\psi^\dagger\psi$?

Hint: Remind that scalars are invariant under general Lorentz transformations, while pseudoscalars receive a sign flip under parity transformations. Vectors transform similar to a four-vector (e.g. a coordinate x^μ), and under parity transformations, there is a sign flip of the spatial components for vectors, but not for axial vectors. For rank-n tensors, each Lorentz index is transformed separately (i.e. again like a four-vector).

(c) **[2P]** Argue if the following terms in a Lagrangian density are allowed, and, if they are not, point out all the reasons why they are not allowed:

(i) $\mathcal{L}_1 = gA(x)\,\bar C(x)\,C(x)$,

(ii) $\mathcal{L}_2 = \dfrac{1}{m}B_\mu(x)\,B^\mu(x)\,\bar C(x)\,C(x)$,

(iii) $\mathcal{L}_3 = \dfrac{g^2}{m}\partial^\mu B^\nu(x)\,A(y)\,\dfrac{\partial A(x)}{\partial t}$,

(iv) $\mathcal{L}_4 = \dfrac{1}{4}g^4 m^4$.

The coupling g is dimensionless, the coupling m is of dimension 1, the scalar and vector fields A and B_μ are each of dimension 1, and the spinor field C is of dimension $\frac{3}{2}$.

(d) **[1P]** Consider a term in the Lagrangian containing a coupling between a scalar field Φ and Dirac spinors ψ,

$$\hat{\mathcal{L}} = \frac{\lambda_{mn}}{m!n!}\left(\bar\psi\psi\right)^m\Phi^n\,,$$

where $m,n \geq 0$, and determine the mass dimension of the coupling constant λ_{mn} by dimensional analysis.

Exercise 3: Quantised complex scalar field **[8P]**

We consider the Lagrangian of a complex scalar field ϕ, given by

$$\mathcal{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi \,.$$

It is common practice in particle physics to express a complex field by two real fields

$$\phi(x) = \frac{1}{\sqrt{2}} \left(\phi_1(x) + i\phi_2(x)\right) \;, \qquad \phi(x)^* = \frac{1}{\sqrt{2}} \left(\phi_1(x) - i\phi_2(x)\right) \,.$$

(a) **[1P]** Express $\mathcal{L}$ through ϕ_1 and ϕ_2. Why do we need the factor $\frac{1}{\sqrt{2}}$?

(b) **[1P]** The Fourier transformation of the real fields ϕ_i is given by

$$\phi_i(x) = \int \mathrm{d}\tilde{k} \left[a_i(k) \exp(-ik \cdot x) + a_i^\dagger(k) \exp(ik \cdot x)\right]$$

where $\mathrm{d}\tilde{k} = \frac{\mathrm{d}^3 k}{(2\pi)^3 2\omega_k}$ with $\omega_k = \sqrt{\vec{k}^2 + m^2}$ and $a_i, a_i^\dagger$ satisfy the commutation relations defined in the lecture. With this we can define

$$a(k) = \frac{1}{\sqrt{2}} \left(a_1(k) + ia_2(k)\right) \;, \qquad b(k) = \frac{1}{\sqrt{2}} \left(a_1(k) - ia_2(k)\right)$$

as well as $a^\dagger$ and $b^\dagger$. Express $\phi(x)$ and $\phi^*(x)$ through $a, a^\dagger, b, b^\dagger$. Interpret these new operators physically.

(c) **[3P]** Derive the commutators of $a, a^\dagger, b, b^\dagger$ through a_i and $a_i^\dagger$.

(d) **[3P]** The Lagrangian, expressed through the fields ϕ_1 and ϕ_2, is invariant under the transformation

$$\phi_1 \to \phi_1' = \phi_1 \cos\vartheta + \phi_2 \sin\vartheta \,, \qquad \phi_2 \to \phi_2' = -\phi_1 \sin\vartheta + \phi_2 \cos\vartheta \,,$$

with a constant parameter $\vartheta \in \mathbb{R}$. Calculate the Noether charge Q for this transformation and express it through a_i as well as a and b. Interpret the results.