Exercise Sheet 5

Exercises:
Dr. Christoph Borschensky (christoph.borschensky@kit.edu)
Dr. Francisco Arco (francisco.arco@kit.edu)
M.Sc. Felix Egle (felix.egle@kit.edu)

Issued: Mon, 27.11.23
Hand-in Deadline: Mon, 04.12.23
Discussion: Wed/Thu, 06/07.12.23

Exercise 1: Pauli-Jordan distribution (I) [7P]

The commutator of a real scalar field $\phi(x)$, evaluated at two different spacetime points x and y,

$$[\phi(x), \phi(y)] \equiv i\Delta(x-y)\,,$$

is also called the *Pauli-Jordan distribution* $\Delta(x-y)$. Using the relations for the commutator $[a(k), a^\dagger(k')]$ of the annihilation and creation operators, the Pauli-Jordan distribution can be written as

$$i\Delta(x-y) = \frac{1}{(2\pi)^3}\int d^4k\, \epsilon(k^0)\, \delta(k^2-m^2)\, e^{-ik\cdot(x-y)} \tag{1.1}$$

with $\epsilon(x) \equiv \theta(x) - \theta(-x)$ and the Heaviside theta function

$$\theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}.$$

(a) **[1P]** Show that the Pauli-Jordan distribution is a Lorentz-invariant quantity. It can thus only depend on Lorentz-invariant quantities itself. Which are those?
Hint: Show first that the integration measure

$$d\tilde{k} = \frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\theta(k^0) = \frac{d^3k}{(2\pi)^3 2\omega_k}$$

with $\omega_k = \sqrt{m^2+\vec{k}^2}$ is Lorentz-invariant. Is $\epsilon(k^0)$ Lorentz-invariant? Again, we only care about orthochronous Lorentz transformations.

(b) **[6P]** Prove the following properties of the Pauli-Jordan distribution, as introduced in the lecture:

(i) $(\Box_x + m^2)\,\Delta(x-y) = (\Box_y + m^2)\,\Delta(x-y) = 0$

(ii) $\Delta(x-y) = -\Delta(y-x)$

(iii) $\Delta(x-y)\Big|_{x^0=y^0} = 0$ (i.e. $\Delta(x-y)$ vanishes for equal times)

(iv) $\Delta(x-y) = 0$ for $(x-y)^2 < 0$ (i.e. $\Delta(x-y)$ vanishes for spacelike distances $x-y$)

(v) $\left(\frac{\partial}{\partial x^0}\Delta(x-y)\right)\Big|_{x^0=y^0} = -\delta^{(3)}(\vec{x}-\vec{y})$.

Hint: Assume that it is always possible to boost a spacelike four-vector $r_\mu = (r^0, \vec{r})^T$ into a reference frame where the temporal component vanishes, i.e. $r_\mu \to r'_\mu = (0, \vec{r}')^T$, where $r^2 = r'^2 = r'_\mu r'^\mu = -\vec{r}'^2 < 0$ is obvious.

Exercise 2: Pauli-Jordan distribution (II) **[8P]**

The momentum integral of the Pauli-Jordan distribution as introduced in exercise 1 can be evaluated explicitly to write $\Delta(r)$ with $r := x - y$ as:

$$\Delta(r) = \frac{m}{4\pi\sqrt{r^2}}\epsilon(r^0)\,\theta(r^2)\,J_1(m\sqrt{r^2}) - \frac{1}{2\pi}\epsilon(r^0)\,\delta(r^2)\,, \tag{2.1}$$

where $\epsilon(x)$ and $\theta(x)$ are defined in exercise 1, and $J_1(x)$ denotes the Bessel function of the first kind. The first term in this expression denotes the case where the spacetime distance r is timelike, i.e. $r^2 > 0$, and the second term denotes lightlike spacetime distances, i.e. $r^2 = 0$.

Note: To avoid confusion: while r^0 in Eq. (2.1) and the following means the temporal component of the four-vector r^μ, r^2 stands for the square, i.e. the invariant product $r^2 = r_\mu r^\mu$ (and not the second spatial component of r).

(a) **[2P]** Show that it is possible to rewrite the integral of Eq. (1.1) of exercise 1 into the following form:

$$\Delta(r) = -\frac{1}{2\pi^2\bar{r}}\int_0^\infty d\bar{k}\,\frac{\bar{k}}{\omega_k}\sin(\bar{k}\bar{r})\sin(\omega_k r^0)\,, \tag{2.2}$$

where we introduced the variables $\bar{r} := |\vec{r}|$, $\bar{k} := |\vec{k}|$, and $\omega_k = \sqrt{m^2 + \bar{k}^2}$.

Hint: Use spherical coordinates with the integration element

$$\int d^3k = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\vartheta \int_0^\infty d\bar{k}\,\bar{k}^2$$

and integrate over the angular variables φ and $\cos\vartheta$.

(b) **[3P]** Consider the case of timelike spacetime distances, $r^2 > 0$, evaluate the integral and express it through the Bessel function of the first kind $J_1(m\sqrt{r^2})$ as in the first term of Eq. (2.1).

Hint: The Bessel function of the first kind $J_1(x)$ is given by the following integral representation:

$$J_1(x) = -\frac{2x}{\pi}\int_1^\infty dt\,\sqrt{t^2-1}\sin(xt) \qquad \text{for } x > 0\,.$$

Since the Pauli-Jordan distribution is Lorentz-invariant, choose a suitable reference frame for r which simplifies the integrand of Eq. (2.2). Remind furthermore that

$$\frac{\sin x}{x} \xrightarrow{x\to 0} 1\,.$$

Distinguish between the cases of $r^0 > 0$ and $r^0 < 0$ to properly track the minus sign in the factor of $\epsilon(r^0)$.

(c) **[3P]** Consider now lightlike spacetime distances, $r^2 = 0$, and evaluate the integral of Eq. (2.2) in this case. The integral will diverge in the *ultraviolet* regime, i.e. for large momenta $\bar{k}$, as denoted by the appearance of the delta distribution in the second term of Eq. (2.1). To simplify the integration, assume thus $\bar{k} \gg m$.

Hint: The one-dimensional delta distribution can be represented in the following integral form:

$$\delta(x - x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\,\cos\left(k(x - x_0)\right)\,.$$

Exercise 3: Quantization of the Complex Scalar Field - Continued **[5P]**

We again look at the Lagrangian for a complex scalar field,

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi,$$

where we again can write ϕ and $\phi^\dagger$ as

$$\begin{aligned}\phi(x) &= \int \mathrm{d}\widetilde{k} \left[a(\vec{k}) e^{-ik\cdot x} + b^\dagger(\vec{k}) e^{ik\cdot x} \right] \\ \phi^\dagger(x) &= \int \mathrm{d}\widetilde{k} \left[b(\vec{k}) e^{-ik\cdot x} + a^\dagger(\vec{k}) e^{ik\cdot x} \right].\end{aligned}$$

As in the lecture (and as you showed on the last sheet) the operators $a, a^\dagger, b, b^\dagger$ fulfill the commutator relations

$$\left[a(\vec{k}), a^\dagger(\vec{k}') \right] = \left[b(\vec{k}), b^\dagger(\vec{k}') \right] = (2\pi)^3 \, 2\omega_k \, \delta(\vec{k} - \vec{k}')$$
$$\text{all other commutators} = 0.$$

(a) **[3P]** Show that the normal ordered 4-momentum operator P_μ can be written as

$$P_\mu = \int \mathrm{d}^3 x :T^0{}_\mu: = \int \mathrm{d}\widetilde{k} \, k_\mu \left[a^\dagger(\vec{k}) \, a(\vec{k}) + b^\dagger(\vec{k}) \, b(\vec{k}) \right] ,$$

in terms of creation and annihilation operators.

(b) **[2P]** In the lecture and in the last exercise sheet we derived the conserved charge

$$Q = \int \mathrm{d}\widetilde{k} \; \left[a^\dagger(\vec{k}) \, a(\vec{k}) - b^\dagger(\vec{k}) \, b(\vec{k}) \right]$$

Show that the commutation relations

$$\begin{aligned}\left[P_\mu, a^\dagger(\vec{k}) \right] &= k_\mu a^\dagger(\vec{k}) \,, & \left[P_\mu, a(\vec{k}) \right] &= -k_\mu a(\vec{k}) \,, \\ \left[P_\mu, b^\dagger(\vec{k}) \right] &= k_\mu b^\dagger(\vec{k}) \,, & \left[P_\mu, b(\vec{k}) \right] &= -k_\mu b(\vec{k}) \,, \\ \left[Q, a^\dagger(\vec{k}) \right] &= a^\dagger(\vec{k}) \,, & \left[Q, a(\vec{k}) \right] &= -a(\vec{k}) \,, \\ \left[Q, b^\dagger(\vec{k}) \right] &= -b^\dagger(\vec{k}) \,, & \left[Q, b(\vec{k}) \right] &= b(\vec{k})\end{aligned}$$

are fulfilled and use them to show that $|a(\vec{k})\rangle = a^\dagger(\vec{k})|0\rangle$ describes a particle with momentum k_μ and charge 1 and $|b(\vec{k})\rangle = b^\dagger(\vec{k})|0\rangle$ describes a particle with momentum k_μ and charge -1.