Exercise Sheet 6

Exercises:
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Exercise 1: Dirac field momentum and charge [8P]

The solution of the Dirac equation can be expanded in plane waves as follows

$$\psi(x) = \int \mathrm{d}\tilde{k} \sum_{\lambda=\pm} \left[a_\lambda(k) u(k,\lambda) e^{-ik\cdot x} + b_\lambda^\dagger(k) v(k,\lambda) e^{ik\cdot x} \right], \quad \text{with} \quad \mathrm{d}\tilde{k} = \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2\omega_k}.$$

Therein, $u(k,\lambda)$ and $v(k,\lambda)$ are Dirac spinors associated with positive and negative energy solutions, respectively. They obey the relations

$$u^\dagger(k,\lambda)u(k,\lambda') = v^\dagger(k,\lambda)v(k,\lambda') = 2\omega_k \delta_{\lambda\lambda'},$$
$$u^\dagger(\bar{k},\lambda)v(k,\lambda') = v^\dagger(\bar{k},\lambda)u(k,\lambda') = 0,$$

where $\bar{k} = (\omega_k, -\vec{k})^T$. At this stage, we leave open whether the $a_\lambda^{(\dagger)}(k)$ and $b_\lambda^{(\dagger)}(k)$ follow commutation or anti-commutation relations.

(a) **[4P]** Show that the components $T^{0\mu}$ of the energy-momentum tensor are given by $T^{0\mu} = \psi^\dagger i\partial^\mu\psi$. Express the four-momentum of the Dirac field

$$P^\mu = \int \mathrm{d}^3x\, T^{0\mu}$$

in terms of $a_\lambda(k), a_\lambda^\dagger(k), b_\lambda(k)$ and $b_\lambda^\dagger(k)$.

(b) **[2P]** Now, express the charge of the Dirac field

$$Q = \int \mathrm{d}^3x\, \overline{\psi}(x)\gamma^0\psi(x)$$

in terms of the coefficients $a_\lambda(k), a_\lambda^\dagger(k), b_\lambda(k)$ and $b_\lambda^\dagger(k)$.

(c) **[2P]** For both subexercises (a) and (b), argue why having anti-commutation relations for b and $b^\dagger$ leads to physically sensible results.

Exercise 2: The Feynman propagator [6P]

On the previous sheet 5, you have derived several properties of the Pauli-Jordan distribution $\Delta(x-y)$, a type of propagator for the scalar field $\phi(x)$. One of its properties was *microcausality*, i.e. the vanishing of $\Delta(x-y)$ for spacelike distances $(x-y)^2 < 0$. This can e.g. also be seen directly from sheet 5, Eq. (2.2), where the integrand vanishes if a suitable reference frame is chosen with the temporal component of the spacelike distance $x^0 - y^0 = r^0 = 0$. In the lecture, you got to know also another type of propagator, the *Feynman propagator* $\Delta_F(x-y)$, which can be written in the following integral representation:

$$i\Delta_F(x-y) = \lim_{\epsilon\to 0^+} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)} \,. \tag{2.1}$$

We will discuss it here, similarly to the Pauli-Jordan propagator, in the case for Hermitian scalar fields $\phi = \phi^\dagger$.

(a) **[3P]** First, show the following properties:

(i) $\Delta_F(x-y)$ is an even function: $\Delta_F(x-y) = \Delta_F(y-x)$.

(ii) $\Delta_F(x-y)$ is the Green's function of the Klein-Gordon equation:

$$\left(\Box_x + m^2\right)\Delta_F(x-y) = \left(\Box_y + m^2\right)\Delta_F(x-y) = -\delta^{(4)}(x-y).$$

(iii) $\Delta_F(x-y)$ is invariant under Poincaré transformations:

$$\Delta_F(x'-y') = \Delta_F(x-y) \qquad \text{for} \qquad x' = \Lambda x + b, \quad y' = \Lambda y + b.$$

(b) **[3P]** We now want to evaluate the integral of $\Delta_F(x-y)$. After performing the k_0 integration in Eq. (2.1) (you do not have to do this), you have seen in the lecture that the Feynman propagator can also be written as (with $r = x - y$):

$$i\Delta_F(r) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[\theta(r^0)e^{-ik\cdot r} + \theta(-r^0)e^{ik\cdot r}\right], \tag{2.2}$$

where $\omega_k = \sqrt{\vec{k}^2 + m^2}$. Since the steps for timelike ($r^2 > 0$) and lightlike ($r^2 = 0$) spacetime distances are analogous (with slightly different Bessel functions) to the case of the Pauli-Jordan propagator, we do not ask you to repeat them here.

Evaluate thus the integral only for *spacelike* distances, $r^2 < 0$, and show that the Feynman propagator, in contrast to the Pauli-Jordan propagator, does *not* vanish, and leads to the following result:

$$i\Delta_F(r)\Big|_{r^2<0} = \frac{m}{4\pi^2\sqrt{-r^2}} K_1^{(1)}(m\sqrt{-r^2})\,,$$

where we introduced the *modified Hankel function*

$$K_1^{(1)}(x) = \frac{1}{2i}\int_{-\infty}^{\infty} dt \frac{te^{ixt}}{\sqrt{t^2+1}} \qquad \text{for } x > 0\,,$$

which is also a type of Bessel function (of the third kind).

Exercise 3: Causality and the spin-statistics theorem **[6P]**

In the previous exercise 2, you found that the Feynman propagator is non-zero for spacelike intervals, which seemed to be at odds with causality. However, in quantum field theories, causality is normally defined over the (anti)commutator of the fields rather than over the propagators. For this exercise, assume that we consider quantized scalar fields of the form

$$\phi^+(x) = \int d\tilde{k}\, e^{ik\cdot x} a(\vec{k})\,, \qquad \phi^-(x) = \int d\tilde{k}\, e^{-ik\cdot x} a^\dagger(\vec{k})\,. \tag{3.1}$$

The operators $a(\vec{k})$, $a^\dagger(\vec{k})$ shall obey the usual algebra

$$\left[a(\vec{k}), a(\vec{k}')\right]_\mp = \left[a^\dagger(\vec{k}), a^\dagger(\vec{k}')\right]_\mp = 0\,,$$
$$\left[a(\vec{k}), a^\dagger(\vec{k}')\right]_\mp = (2\pi)^3\, 2\omega_k\, \delta(\vec{k} - \vec{k}')\,,$$

where the index '$-$' denotes the commutator, while the index '$+$' denotes the anticommutator, and the index '$\mp$' implies that we leave this choice open. Causality requires that for two field operators that are separated by a spacelike interval, we find

$$[\phi(x), \phi(y)]_\mp = 0 \qquad \text{for } (x-y)^2 < 0\,. \tag{3.2}$$

(a) **[2P]** Discuss either mathematically or geometrically (e.g. with the help of a Minkowski diagram) why the vanishing (anti)commutator for spacelike intervals is an argument of causality, i.e. why a "measurement" of $\phi(x)$ should not influence another "measurement" $\phi(y)$ where x and y are separated by a spacelike interval, and why this does not have to hold for timelike intervals, where the (anti)commutator may be non-zero.

(b) **[2P]** Show that the fields $\phi^+(x)$ and $\phi^-(x)$ of Eq. (3.1) do not obey the (anti)commutator relation of Eq. (3.2) for spacelike intervals.

Hint: Calculate Eq. (3.2) for spacelike intervals $r^2 \equiv (x-y)^2 < 0$ for all combinations of $\phi^+(x)$ and $\phi^-(x)$. You can use the results from exercise 2.

In order to restore causality, we define new fields

$$\phi_\lambda(x) = \phi^+(x) + \lambda\phi^-(x)\ , \quad \phi_\lambda^\dagger(x) = \phi^-(x) + \lambda^*\phi^+(x)$$

with a complex parameter λ.

(c) **[2P]** Calculate Eq. (3.2) for all possible combinations of ϕ_λ and $\phi_\lambda^\dagger(x)$. What is the value of λ if we require causality for both fields? Do you have to choose commutators or anticommutators to restore causality?

Note: The connection between the spin of the fields and their algebra is called the spin-statistics theorem. In part (c), you will see that the requirement of causality automatically leads to the correct spin-statistics of scalar fields.