Theoretical Particle Physics I

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Exercise Sheet 7

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Exercise 1: Polarizations and propagator for massive vector boson [8P]

The Lagrangian for a vector boson with mass $m \neq 0$, described by the field V_μ, is given by:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} V_\mu V^\mu$$

with $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. You discussed this Lagrangian already on exercise sheet 3, and while deriving the equations of motion, found that for the massive vector field, the Lorentz gauge condition $\partial_\mu V^\mu = 0$ is automatically fulfilled.

(a) **[4P]** To quantise the massive vector field, we introduce as usual a Fourier decomposition of the form:

$$V_\mu(x) = \int \mathrm{d}\tilde{k} \sum_{r=0}^{3} \left(\epsilon_\mu^{(r)}(k)\, a^{(r)}(k)\, e^{-ik\cdot x} + \epsilon_\mu^{(r)*}(k)\, a^{(r)\dagger}(k)\, e^{ik\cdot x} \right) .$$

A priori, this includes four polarization vectors $\epsilon_\mu^{(r)}(k)$. Due to $\partial_\mu V^\mu = 0$, only three linearly independent polarization vectors survive. Show that a convenient basis for these polarization vectors, in the reference frame with $\vec{k} = (0, 0, |\vec{k}|)^T$, is given by

$$\epsilon_\mu^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} , \qquad \epsilon_\mu^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} , \qquad \epsilon_\mu^{(3)} = \frac{1}{m} \begin{pmatrix} |\vec{k}| \\ 0 \\ 0 \\ \omega_k \end{pmatrix}$$

for the three physical polarization vectors, which are orthogonal to the unphysical polarization vector $\epsilon_\mu^{(0)} = k_\mu / m$. The physical polarization vectors obey the orthonormality condition

$$\epsilon^{(r),\mu}(k) \epsilon_\mu^{(s)*}(k) = -\delta_{rs} .$$

Furthermore, derive the form of the completeness relation,

$$\sum_{r=1}^{3} \epsilon_\mu^{(r)}(k) \epsilon_\nu^{(r)*}(k) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} ,$$

from general considerations, i.e. by writing down all possible Lorentz tensors and using the properties of the $\epsilon_\mu^{(r)}(k)$, and show that the given polarization vectors above fulfill this relation.

(b) **[4P]** We now impose standard bosonic commutation relations for the surviving operators. They read

$$[a^{(r)}(k), a^{(s)}(k')] = [a^{(r)\dagger}(k), a^{(s)\dagger}(k')] = 0,$$
$$[a^{(r)}(k), a^{(s)\dagger}(k')] = \delta_{rs} 2\omega_k (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \, .$$

Verify, in a similar manner to what was shown in the lecture for the scalar field, that the Feynman propagator of the massive vector boson takes the form

$$\langle 0 | T V^\mu(x) V^\nu(y) | 0 \rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\varepsilon} \left(-g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2} \right) e^{-ik\cdot(x-y)} \, ,$$

where T is the time-ordering operator for bosonic fields (i.e. the same as in the scalar case).

Hint: Evaluate the k^0 integral on the right-hand side of the equation (treat the poles of the integrand carefully), and show that it corresponds to the expression on the left-hand side.

Note: As the photon has only two rather than three physical degrees of freedom, the limit $m \to 0$ of this propagator is not well-defined and does not yield the photon propagator.

Exercise 2: Electromagnetic stress-energy tensor **[4P]**

The Lagrangian for a massless vector field is given by:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

Derive Maxwell's equations as the Euler-Lagrange equations and calculate the energy-momentum tensor $T^{\mu\nu}$ given by Noether's Theorem. Note that this does not yield a symmetric tensor. We can, however, use the fact that

$$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu}$$

is also a conserved quantity, if $K^{\lambda\mu\nu}$ is antisymmetric in its first two indices (convince yourself of this fact). Show that this construction with

$$K^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu \, ,$$

leads to a symmetric energy tensor $\hat{T}^{\mu\nu}$ and also yields the standard formula for the electromagnetic energy and momentum densities, i.e.

$$\hat{T}^{00} = \mathcal{E} = \frac{1}{2}(E^2 + B^2) \, , \qquad \hat{T}^{0i} = S^i = (\vec{E} \times \vec{B})^i \, ,$$

by identifying $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$.

Hint: Use the equation of motion to simplify your calculation. Beware of upper and lower indices and resulting minus signs.

Exercise 3: Vacuum of the Gupta-Bleuler photon **[8P]**

In the Gupta-Bleuler formalism of the free photon field, the most general vacuum state reads

$$|\varphi\rangle = \sum_{n=0}^{\infty} C_n |\varphi_n\rangle \, .$$

The states $|\varphi_n\rangle$ do not include transverse photons, but exactly n scalar and longitudinal photons. The additional condition

$$(a^{(3)}(k) - a^{(0)}(k))|\varphi_n\rangle = 0$$

makes these states physical. We moreover choose $|\varphi_0\rangle = |0\rangle$.

(a) **[3P]** Show that the most general form of $|\varphi_1\rangle$ is given by

$$|\varphi_1\rangle = \int d\tilde{q}\, f(q) \left(a^{(3)\dagger}(q) - a^{(0)\dagger}(q)\right) |0\rangle \, .$$

Hint: Use the ansatz

$$|\varphi_1\rangle = \int d\tilde{q} \sum_{r=0,3} a^{(r)\dagger}(q) f^{(r)}(q) |0\rangle$$

with an arbitrary function $f^{(r)}(q)$.

(b) **[5P]** Show that the expectation value of the photon field in the above general vacuum state corresponds to a pure gauge, i.e.

$$\langle\varphi|A_\mu(x)|\varphi\rangle = \partial_\mu \Lambda(x) \, ,$$

where, using the explicit polarization vectors $\epsilon_\mu^{(0)}(k) = n_\mu$ and $\epsilon_\mu^{(3)}(k) = \frac{k_\mu}{k\cdot n} - n_\mu$ (with n_μ an arbitrary 4-vector for which $n \cdot k \neq 0$), the function $\Lambda(x)$ is given by

$$\Lambda(x) = \int \frac{d\tilde{k}}{k \cdot n} \, 2\mathrm{Re}\left(i C_0^* C_1 e^{-ik\cdot x} f(k)\right) \, .$$

Therein, $f(k)$ is identical to the one in subexercise (a). The function $\Lambda(x)$ fulfills $\Box\Lambda(x) = 0$ and can be chosen arbitrarily through the choice of the corresponding vacuum state $|\varphi\rangle$.

Hint: First, show that

$$\langle\varphi_n|N A_\mu(x)|\varphi_{n-1}\rangle = \langle\varphi_n|A_\mu(x)|\varphi_{n-1}\rangle$$

with

$$N = \int d\tilde{k}(a^{(3)\dagger}(k)a^{(3)}(k) - a^{(0)\dagger}(k)a^{(0)}(k))$$

the operator that counts longitudinal and scalar photons. Thus it yields $\langle\varphi_n|A_\mu(x)|\varphi_{n-1}\rangle = 0$ for $n \neq 1$.