Exercise Sheet 8

Exercises:
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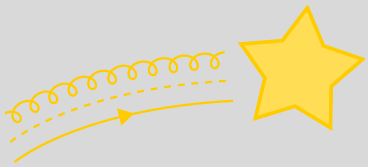
Note that due to the Christmas holidays, the deadline for this sheet will be in the new year.



We wish you all happy holidays and a good start into the new year 2024!

Exercise 1: Propagator of a massless vector boson [8P]

The Lagrangian for a massless vector field A_μ is given as:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor.

(a) **[3P]** The Feynman propagator $i\Delta_F^{\mu\nu}(x-y) := \langle 0|T(A^\mu(x)A^\nu(y))|0\rangle$ is the Green's function of the equation of motion for A_μ:

$$\left(-g_{\mu\nu}\Box + \partial_\mu\partial_\nu\right) i\Delta_F^{\nu\rho}(x) = -\delta_\mu^\rho \delta^{(4)}(x) \,. \tag{1.1}$$

Using Eq. (1.1), attempt to obtain an expression for $i\Delta_F^{\nu\rho}(x)$. At which step do you run into a problem, and why?

Hint: Do a Fourier transformation of the equation of motion, and try to solve for $\tilde{\Delta}_F^{\nu\rho}(k)$, where the latter is the Fourier transform of $\Delta_F^{\nu\rho}(x)$. Make sure to be consistent with the Lorentz indices.

(b) **[1P]** Add a gauge-fixing term

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

with the gauge-fixing parameter ξ to $\mathcal{L}$ and quickly rederive the equation of motion for the field A_μ.

(c) **[4P]** Write down the analogous equation to Eq. (1.1) for the Feynman propagator $\Delta_F^{\mu\nu}(x)$ as in part (a), now including also the gauge-fixing term of part (b), and derive the solution for $\Delta_F^{\mu\nu}(x)$:

$$\Delta_F^{\mu\nu}(x) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} \left[-g^{\mu\nu} + (1-\xi)\,\frac{k^\mu k^\nu}{k^2 + i\epsilon}\right]$$

Hint: Again, do a Fourier transformation of the equation as in part (a). Make the ansatz:

$$\tilde{\Delta}_F^{\mu\nu}(k) = aX^{\mu\nu} + bY^{\mu\nu}\,,$$

and determine the coefficients a and b. What are the two different tensors $X^{\mu\nu}$ and $Y^{\mu\nu}$ of rank 2 that you can construct with your available building blocks? Note that in this derivation, you will not get the $i\epsilon$ terms in the denominator of the propagator. Thus, simply assume them to be implied.

Exercise 2: Causality of the Green's function **[7P]**

The solution of the differential equation $(\Box_x + m^2)G(x-y) = -\delta^{(4)}(x-y)$, i.e. the Green's function of the Klein-Gordon equation, can be written in the form

$$G(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2}e^{-ik\cdot(x-y)}\,.$$

You have already learned in the lecture and the previous exercises that the Feynman propagator $i\Delta_F(x-y)$ is one of the Green's functions of the Klein-Gordon equation. There exist, however, other possibilities, which we want to discuss in this exercise.

(a) **[2P]** Performing the integration over k_0 the integrand has a pole at $k_0 = \pm\sqrt{\vec{k}^2+m^2}$. There are four potential integration paths in the complex k_0 plane. The choice of the four options can be made explicit by infinitesimal shifts of the poles by $\pm i\epsilon$ (with $\epsilon > 0$). Which paths correspond to the following Green's functions?

$$\begin{aligned}
\Delta_{\text{ret}}(x-y) &= \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{1}{(k_0+i\epsilon)^2-\vec{k}^2-m^2}\\
\Delta_{\text{av}}(x-y) &= \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{1}{(k_0-i\epsilon)^2-\vec{k}^2-m^2}\\
\Delta_F(x-y) &= \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{1}{k^2-m^2+i\epsilon}\\
\Delta_D(x-y) &= \int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-y)}\frac{1}{k^2-m^2-i\epsilon}
\end{aligned}$$

Hint: Determine the poles in terms of $\omega_k = \sqrt{\vec{k}^2+m^2}$ and ϵ and draw the four poles in the complex k_0 plane.

(b) **[5P]** Show that the four functions can be expressed through the real (quantized) fields ϕ and the vacuum state $|0\rangle$ as follows

$$\begin{aligned}
i\Delta_{\text{ret}}(x-y) &= \theta(x_0-y_0)\langle 0|[\phi(x),\phi(y)]|0\rangle\\
i\Delta_{\text{av}}(x-y) &= -\theta(y_0-x_0)\langle 0|[\phi(x),\phi(y)]|0\rangle\\
i\Delta_F(x-y) &= \langle 0|T\phi(x)\phi(y)|0\rangle\\
i\Delta_D(x-y) &= -\theta(x_0-y_0)\langle 0|\phi(y)\phi(x)|0\rangle - \theta(y_0-x_0)\langle 0|\phi(x)\phi(y)|0\rangle
\end{aligned}$$

using the time-ordering operator T given by

$$T\phi(x)\phi(y) = \begin{cases} \phi(x)\phi(y), & \text{if } x_0 > y_0 \\ \phi(y)\phi(x), & \text{if } x_0 < y_0 \end{cases} .$$

It moreover yields $\theta(t) = 1$ for $t \geq 0$, otherwise 0. What are the consequences for causality for the four Green's functions?

Exercise 3: Interaction picture **[5P]**

As we are about to discuss interacting theories in the lecture, let us recap some concepts regarding the time evolution of states and operators in quantum mechanics. In the Schrödinger picture, the time evolution of a state is described by the Schrödinger equation

$$i\frac{d}{dt}|\psi(t)\rangle^S = H|\psi(t)\rangle^S$$

with the Hamilton operator H, whereas an operator O^S itself is constant in time. For the interaction picture we split the Hamilton operator into two parts,

$$H = H_0 + H_I ,$$

where H_0 describes free particles and H_I the interactions among them. The unitary time evolution operator $U_0(t, t_0)$, defined by $U_0(t, t_0) = e^{-iH_0(t-t_0)}$, yields the time evolution of the states in the Schrödinger picture in the case of free particles, $H_I = 0$. It is used to define the transformation from the Schrödinger to the interaction picture through

$$|\psi(t)\rangle^I = U_0^\dagger(t, t_0)|\psi(t)\rangle^S .$$

(a) **[3P]** From the equivalence of the matrix element

$${}^I\langle\psi(t)|O^I(t)|\psi(t)\rangle^I = {}^S\langle\psi(t)|O^S|\psi(t)\rangle^S ,$$

find the form of the operator $O^I(t)$ in the interaction picture depending on O^S. What do you obtain for the case of the free Hamiltonian, $O^S = H_0$? Determine also the time evolution of the operator $O^I(t)$, i.e. $i\frac{d}{dt}O^I(t)$, and show that

$$i\frac{d}{dt}|\psi(t)\rangle^I = H_I^I(t)|\psi(t)\rangle^I ,$$

where $H_I^I(t)$ denotes the operator H_I, i.e. the interaction Hamiltonian, in the interaction picture. How does the solution for $|\psi(t)\rangle^I$ look like?

(b) **[2P]** Consider the Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$ with

$$\mathcal{L}_0 = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{m^2}{2}\phi^2 , \qquad \mathcal{L}_I = -\frac{\lambda}{4}\phi^4 .$$

Determine H_0 and H_I. Which differential equation describes the time evolution of the field operators $\phi^I(t)$ in the interaction picture? Remind the Fourier decomposition of $\phi^I(t)$ and $:H_0:= \int d\tilde{k}\, k_0\, a^\dagger a$ to confirm that the $\phi^I(t)$ are a solution of this equation.