Theoretical Particle Physics I

Prof. Dr. Milada Margarete Mühlleitner **WS 2023/24**

Exercise Sheet 9

Exercises:
Dr. Christoph Borschensky (`christoph.borschensky@kit.edu`)
Dr. Francisco Arco (`francisco.arco@kit.edu`)
M.Sc. Felix Egle (`felix.egle@kit.edu`)

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Exercise 1: Wick's theorem [8P]

In the calculation of S-matrix elements, we come across general n-point correlation functions,

$$\langle 0|T\phi(x_1)\phi(x_2)\cdots\phi(x_n)|0\rangle$$

with the n fields evaluated at different spacetime points $x_1, \ldots, x_n$, and the time-ordering operator T as introduced in the lecture. Wick's theorem allows us to express these correlation functions in terms of two-point correlation functions which we already computed and know as the Feynman propagators, $\langle 0|T\phi(x_i)\phi(x_j)|0\rangle = i\Delta_F(x_i - x_j)$.

In this exercise, we want to study Wick's theorem for a real scalar field ϕ. In the following, we split up ϕ into two parts corresponding to positive (ϕ^+) and negative frequencies (ϕ^-),

$$\phi^+(x) = \int d\tilde{k}\, a(k)e^{-ik\cdot x}\,, \qquad \phi^-(x) = \int d\tilde{k}\, a^\dagger(k)e^{ik\cdot x}\,, \qquad \text{so that} \quad \phi(x) = \phi^+(x) + \phi^-(x)\,.$$

We use the following notation for a Wick contraction:

$$\phi(x)\phi(y) := \langle 0|T\phi(x)\phi(y)|0\rangle = i\Delta_F(x-y)\,.$$

(a) **[1P]** Argue why $\langle 0|{:}O{:}|0\rangle = 0$ for a normal-ordered operator O, where O could e.g. be a product of an arbitrary number of field operators $\phi(x)$.

(b) **[1P]** Verify that

$$\phi(x)\phi(y) = {:}\phi(x)\phi(y){:} + \left[\phi^+(x), \phi^-(y)\right].$$

Use this to show explicitly that

$$T\phi(x)\phi(y) = {:}\phi(x)\phi(y){:} + f(x,y)\,,$$

where $f(x,y)$ is a c-number. Give the explicit form of $f(x,y)$ in terms of commutators and show that $f(x,y) = \phi(x)\phi(y) = i\Delta_F(x-y)$.

(c) **[2P]** Derive, in a similar way to the case of two fields, the expression for three fields as shown in the lecture:

$$T\phi(x_1)\phi(x_2)\phi(x_3) = {:}\phi(x_1)\phi(x_2)\phi(x_3){:} + {:}\phi(x_1){:}\,\phi(x_2)\phi(x_3) + {:}\phi(x_2){:}\,\phi(x_1)\phi(x_3) + {:}\phi(x_3){:}\,\phi(x_1)\phi(x_2)$$

Hint: Start with the product of fields on the left-hand side and write again all fields in terms of the $\phi^+(x)$ and $\phi^-(x)$.

(d) **[2P]** Prove Wick's theorem for n fields, where, for simplicity, we write $\phi_i \equiv \phi(x_i)$:

$$
\begin{aligned}
T\phi_1\cdots\phi_n = {}&{:}\phi_1\cdots\phi_n{:} \\
&+ {:}\wick{\c\phi_1\c\phi_2}\cdots\phi_n{:} + {:}\wick{\c\phi_1\phi_2\c\phi_3}\cdots\phi_n{:} + \ldots \quad \boxed{\text{“single” contractions}} \\
&+ {:}\wick{\c1\phi_1\c1\phi_2\c2\phi_3\c2\phi_4}\cdots\phi_n{:} + {:}\wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4}\cdots\phi_n{:} + \ldots \quad \boxed{\text{“double” contractions}} \\
&+ \ldots \\
&+ {:}\wick{\c1\phi_1\c1\phi_2}\cdots\wick{\c2\phi_{n-1}\c2\phi_n}{:} + \ldots \quad \boxed{\text{“full” contractions}}
\end{aligned}
$$

The terms in the boxes stand for all permutations with one pair of fields being contracted (“single”), two pairs being contracted (“double”), ..., all fields being contracted (“full”). Note that for an odd number n of fields, in the terms of the last line, there will always be one uncontracted field left over (see e.g. the case for $n = 3$ in part (c)).

Hint: Prove the equation via induction. Assume for simplicity that all fields are already ordered in time, i.e. $x_1^0 > x_2^0 > \ldots > x_n^0$, so that $T(\phi_1\cdots\phi_n) = \phi_1\cdots\phi_n$ and therefore in particular $T(\phi_1\cdots\phi_n) = \phi_1 T(\phi_2\cdots\phi_n)$. If it makes it easier for you, you can also assume that n is even. You might furthermore find the following relation useful to relate the commutator between products of operators to a sum of commutators of two operators only:

$$[A_1, A_2\cdots A_m] = \sum_{i=2}^{m} A_2\cdots A_{i-1}[A_1, A_i]A_{i+1}\cdots A_m \qquad \text{for } m \geq 3\,,$$

which is the generalisation of $[A, BC] = [A, B]C + B[A, C]$. As the commutator between two fields ϕ_i and ϕ_j are c-numbers (i.e. no operators), it also holds that

$$[\phi_1, {:}\phi_2\cdots\phi_n{:}] = {:}[\phi_1, \phi_2\cdots\phi_n]{:}\,,$$

i.e. the normal ordering can be considered after evaluating the commutator. Remind that you can split up the field $\phi_i = \phi_i^+ + \phi_i^-$, and that $[\phi_i^+, \phi_j] = [\phi_i^+, \phi_j^-]$.

Note: Since the Wick contraction $\wick{\c\phi(x)\c\phi(y)}$ is a c-number, i.e. not a operator quantity, we can freely commute it with the uncontracted or other contracted field operators and e.g. write:

$$ {:}\wick{\c\phi_1\phi_2\c\phi_3\phi_4}{:} = {:}\phi_2\phi_4{:}\,\wick{\c\phi_1\c\phi_3} = {:}\phi_2\phi_4{:}\,\langle 0|T\phi_1\phi_3|0\rangle \qquad \text{or} \qquad {:}\wick{\c1\phi_1\c2\phi_2\phi_3\c1\phi_4\phi_5\c2\phi_6}{:} = {:}\phi_3\phi_5{:}\,\wick{\c1\phi_1\c1\phi_4\c2\phi_2\c2\phi_6} $$

(e) **[2P]** Which of the terms of part (d) are relevant if we take the vacuum expectation value of the expression, $\langle 0|T\phi_1\ldots\phi_n|0\rangle$? What do you get for $n = 3$ and $n = 4$, or, in general, odd and even n? Can you interpret the result for $n = 4$ graphically?

Exercise 2: Feynman rules in ϕ^4 theory **[12P]**

As a practical application of Wick's theorem we can derive Feynman rules, i.e. diagrammatic rules to construct diagrams and scattering processes. Let us in the following study the Lagrangian of a real scalar field ϕ with a ϕ^4 interaction term:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{m^2}{2}\phi^2 + \mathcal{L}_I \qquad \text{with} \qquad \mathcal{L}_I = -\frac{\lambda}{4!}\phi^4\,, \tag{2.1}$$

where m is the mass of ϕ and λ is a real dimensionless coupling constant. We now add this interaction term to the time-ordered correlation functions,

$$\langle 0|T\phi(x_1)\dots\phi(x_n)\exp\left[i\int d^4z\,\mathcal{L}_I(z)\right]|0\rangle\,. \tag{2.2}$$

We cannot calculate these correlation functions exactly anymore, so we have to do perturbation theory, under the assumption that the coupling constant λ is small, $\lambda \ll 1$.

(a) **[3P]** Let us first of all discuss the case for $n = 0$, $\langle 0|T\exp\left[i\int d^4z\,\mathcal{L}_I(z)\right]|0\rangle$. This correlation function describes vacuum diagrams that are not connected to any external lines, and they therefore do not contribute to scattering processes. Expand the correlation function up to $\mathcal{O}(\lambda^2)$ (i.e. neglect terms of $\mathcal{O}(\lambda^3)$ and higher). Evaluate the resulting terms with Wick's theorem and show that you obtain the following expressions:

$$\begin{aligned}\langle 0|T\exp\left[i\int d^4z\,\mathcal{L}_I(z)\right]|0\rangle = 1 + i\frac{\lambda}{8}\int d^4z\,[\Delta_F(z-z)]^2 \\ -\frac{\lambda^2}{128}\int d^4z_1\int d^4z_2\,[\Delta_F(z_1-z_1)\Delta_F(z_2-z_2)]^2 \\ -\frac{\lambda^2}{16}\int d^4z_1\int d^4z_2\,[\Delta_F(z_1-z_2)]^2\Delta_F(z_1-z_1)\Delta_F(z_2-z_2) \\ -\frac{\lambda^2}{48}\int d^4z_1\int d^4z_2\,[\Delta_F(z_1-z_2)]^4 + \mathcal{O}(\lambda^3)\,.\end{aligned}$$

Count all possible contractions properly in order to get the correct numerical prefactors. Try to interpet the $\mathcal{O}(\lambda)$ and $\mathcal{O}(\lambda^2)$ terms graphically. Denote each factor of a Feynman propagator by a line, and each coordinate z, z_1, z_2 by a point. It is possible that a diagram corresponding to one term can consist of several *disconnected* pieces.

(b) **[3P]** We now focus on the case with two fields, $n = 2$, and expand Eq. (2.2) up to $\mathcal{O}(\lambda)$. The first term of the expansion is nothing more than the plain propagator $i\Delta_F(x-y)$. Use Wick's theorem to show that the second term of $\mathcal{O}(\lambda)$ results in

$$\begin{aligned}\langle 0|T\phi(x)\phi(y)\left[\frac{-i\lambda}{4!}\int d^4z\phi^4(z)\right]|0\rangle = -\frac{\lambda}{8}\Delta_F(x-y)\int d^4z\Delta_F(z-z)\Delta_F(z-z) \\ -\frac{\lambda}{2}\int d^4z\Delta_F(x-z)\Delta_F(y-z)\Delta_F(z-z)\end{aligned}$$

Again, find a pictorial representation (i.e. a Feynman diagram) of the terms. Similarly to the case for the vacuum diagrams, you can get disconnected pieces.

(c) **[1P]** The previous discussion was carried out in position space. However, Feynman rules turn out to be simpler in momentum space. For this purpose we use the well-known representation of the propagator in momentum space

$$i\Delta_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle = \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}\,,$$

such that we associate a momentum p (and its direction) with each line in a Feynman diagram. Insert this representation of the propagator into the last term of the previous subexercise. Show that the integration over z results in momentum conservation at the vertex z.

(d) **[4P]** We finally consider $n = 4$, a correlation function with four external fields. Split the vacuum expectation value of Eq. (2.2), again expanded up to $\mathcal{O}(\lambda)$, into combinations of propagators using Wick's theorem. Draw all Feynman diagrams, both connected and disconnected.

(e) **[1P]** Use the previous subexercises to motivate that the Feynman rule in momentum space for the vertex in combination with momentum conservation is given by

$$(-i\lambda)(2\pi)^4\delta^{(4)}(p_1 + p_2 - p_3 - p_4)\,.$$