Theoretical Particle Physics I

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Exercise Sheet 10

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Exercise 1: S-matrix elements for ϕ^4-theory **[9P]**

On the previous sheet, we have discussed time-ordered correlation functions of the type

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)\exp\left[i\int d^4z\,\mathcal{L}_I(z)\right]|0\rangle\,. \tag{1.1}$$

These correlation functions capture the interaction content of our theory, described by the interaction Lagrangian $\mathcal{L}_I$. In the lecture, you learned about the S-matrix elements

$$\langle f|S|i\rangle\,, \tag{1.2}$$

where the initial $|i\rangle$ and final $\langle f|$ states are asymptotic (in the sense of the time $t\to-\infty$ and $t\to+\infty$) multi-particle states. The S-matrix elements describe transition amplitudes of e.g. a scattering process from an initial to a final state.

The connection between Eqs. (1.1) and (1.2), i.e. between a quantity which we can calculate (correlation function) and a transition amplitude for an experimentally measurable process (S-matrix element), is quite complicated and technical, and goes by the name of *LSZ reduction formula*[1]. We do not want to describe this formula here, but simply use it so that we can calculate $\langle f|S|i\rangle$ as introduced in the lecture, in a similar way to the correlation functions:

$$\langle f|S|i\rangle = \langle f|T\left[\exp\left(i\int d^4z\,\mathcal{L}_I(z)\right)\right]|i\rangle\,, \tag{1.3}$$

with the time-ordering operator T and the multi-particle initial and final states, defined as

$$|i\rangle = |\vec{p}_1,\ldots,\vec{p}_{n_i}\rangle = a^\dagger(\vec{p}_1)\cdots a^\dagger(\vec{p}_{n_i})|0\rangle \quad \text{and analogously} \quad \langle f| = \langle 0|a(\vec{k}_1)\cdots a(\vec{k}_{n_f})\,,$$

with $p_1,\ldots,p_{n_i}$ the n_i initial momenta and $k_1,\ldots,k_{n_f}$ the n_f external ones. Let us for now stick to the simple example of ϕ^4-theory with

$$\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4\,.$$

We begin by discussing the scattering process of two scalar fields $\phi\phi\to\phi\phi$, so $n_i=n_f=2$.

(a) **[2P]** First, determine $\langle f|i\rangle$, i.e. the trivial term in the expansion of the S-matrix of Eq. (1.3), and express it in terms of delta distributions. Can you represent the resulting terms graphically?

[1] named after the authors Lehmann, Symanzik, and Zimmermann

(b) **[1P]** Expand Eq. (1.3) up to $\mathcal{O}(\lambda)$ and use Wick's theorem to rewrite the time-ordered product as terms consisting of normal-ordered operators and Wick contractions. You should obtain

$$\langle f|S|i\rangle\Big|_{\mathcal{O}(\lambda)} = -i\frac{\lambda}{4!}\int d^4z\,\Big[\langle f|{:}\phi(z)\phi(z)\phi(z)\phi(z){:}|i\rangle + 6\,\wick{\c\phi(z)\c\phi(z)}\langle f|{:}\phi(z)\phi(z){:}|i\rangle$$
$$+\,3\,\wick{\c1\phi(z)\c1\phi(z)\c2\phi(z)\c2\phi(z)}\langle f|i\rangle\Big]\,.$$

Note: Contrary to the correlation functions, the S-matrix elements do not vanish if several fields are left uncontracted due to the annihilation and creation operators in $|i\rangle$ and $\langle f|$.

(c) **[3P]** Evaluate the second term of part (b),

$$-6i\frac{\lambda}{4!}\int d^4z\,\wick{\c\phi(z)\c\phi(z)}\langle f|{:}\phi(z)\phi(z){:}|i\rangle\,.$$

Introduce again $\phi^+(x)$ and $\phi^-(x)$ as defined in exercise 1 of sheet 9 to simplify the evaluation of the normal-ordered product and use the definitions for the contractions with the creation and annihilation operators of the external states:

$$\wick{\c\phi^+(x)\c a^\dagger(\vec{p})} = e^{-ip\cdot x} \qquad \text{and} \qquad \wick{\c a(\vec{p})\c\phi^-(x)} = e^{ip\cdot x}\,.$$

Find a graphical representation of this term. Does it describe a scattering process?

(d) **[2P]** Evaluate now the first term of part (b),

$$-i\frac{\lambda}{4!}\int d^4z\,\langle f|{:}\phi(z)\phi(z)\phi(z)\phi(z){:}|i\rangle\,,$$

and show that you obtain:

$$-i\lambda(2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2)\,.$$

Again, represent this term graphically as a diagram, and discuss why only this *fully connected* diagram contributes to the scattering process.

Note: By *fully connected*, we mean that each endpoint of the diagram has to be connected.

(e) **[1P]** We take a quick look at the more complicated $2\to 4$ process $\phi\phi\to\phi\phi\phi\phi$ with $n_i=2$ and $n_f=4$. What do you now get for $\langle f|i\rangle$ (it is enough to describe your answer in words)? Can you draw the 10 different *fully connected* diagrams that contribute to this process at $\mathcal{O}(\lambda^2)$?

Exercise 2: Mandelstam variables [4P]

We consider a $2\to 2$ scattering process. We denote the momentum and mass of each particle in the initial state ($i=1,2$, ingoing momenta) and final state ($i=3,4$, outgoing momenta) by p_i and m_i, respectively. All four particles are on-shell, i.e. $p_i^2=m_i^2$. Thus we have the energy-momentum conservation

$$p_1+p_2=p_3+p_4\,.$$

We define the Mandelstam variables

$$s = (p_1 + p_2)^2 , \qquad t = (p_1 - p_3)^2 , \qquad u = (p_1 - p_4)^2 .$$

(a) **[2P]** How many different scalar products can you build with the given momenta? Show that you can write all of them in terms of s, t, u and the masses m_i.

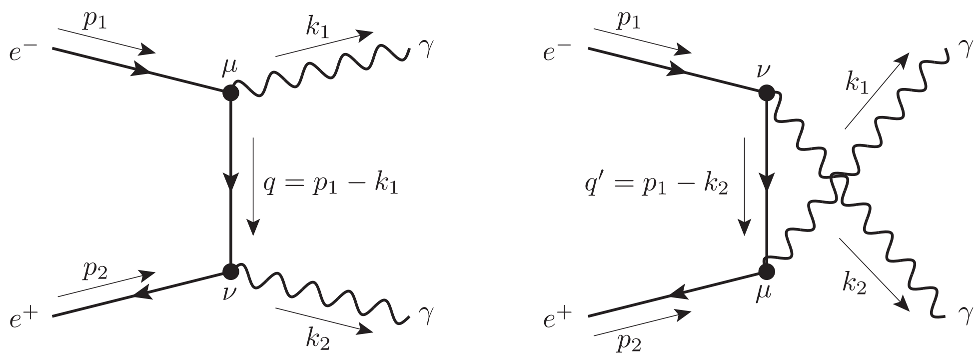
(b) **[2P]** Show that $s + t + u = \sum_{i=1}^{4} m_i^2$ by using four-momentum conservation.

Exercise 3: Pair annihilation in QED **[7P]**

In the lecture, you discussed the annihilation of an electron-positron pair into two photons,

$$e^-(p_1, s_1) + e^+(p_2, s_2) \to \gamma^\mu(k_1, \lambda_1) + \gamma^\nu(k_2, \lambda_2) ,$$

where p_1, p_2 and s_1, s_2 denote the ingoing momenta and spins of the initial-state electron and positron, respectively, and k_1, k_2 as well as λ_1, λ_2 denote the outgoing momenta and polarisations of the final-state photons. The two diagrams for this process are:



The total amplitude can be written as a sum, $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$, where $\mathcal{M}_1$ was given in the lecture as

$$\mathcal{M}_1 = \bar{v}(p_2)(-ie\gamma^\nu)\frac{i}{\not{q} - m + i\epsilon}(-ie\gamma^\mu)u(p_1)\epsilon_\mu^{(\lambda_1)*}(k_1)\epsilon_\nu^{(\lambda_2)*}(k_2)$$

with $\not{q} = q_\mu\gamma^\mu$ and $q = p_1 - k_1$. Both the electron and the positron have the same mass m with $p_1^2 = p_2^2 = m^2$, while the photons are massless, $k_1^2 = k_2^2 = 0$. The necessary Feynman rules of QED can be found in the lecture notes.

(a) **[3P]** Write down the expression for the second diagram, $\mathcal{M}_2$, and show that the sum $\mathcal{M}$ can be written as:

$$\mathcal{M} = -ie^2\epsilon_\mu^{(\lambda_1)*}(k_1)\epsilon_\nu^{(\lambda_2)*}(k_2)\bar{v}(p_2)\left[\frac{\gamma^\nu\not{k}_1\gamma^\mu - 2p_1^\mu\gamma^\nu}{2p_1\cdot k_1} + \frac{\gamma^\mu\not{k}_2\gamma^\nu - 2p_1^\nu\gamma^\mu}{2p_1\cdot k_2}\right]u(p_1)$$

Hint: Use $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ as well as the Dirac equations $(\not{p} - m)u(p) = 0$ and $\bar{v}(p)(\not{p} + m) = 0$.

(b) **[4P]** With $\mathcal{M}^{\mu\nu}$ defined via $\mathcal{M} = \epsilon_\mu^{(\lambda_1)*}(k_1)\epsilon_\nu^{(\lambda_2)*}(k_2)\mathcal{M}^{\mu\nu}$, show the *Ward identity*:

$$k_{1,\mu}\mathcal{M}^{\mu\nu} = k_{2,\nu}\mathcal{M}^{\mu\nu} = 0 .$$

Note: The Ward identity is a consequence of the gauge invariance of the amplitude.