Theoretical Particle Physics I

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Exercise Sheet 11

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Exercise 1: Pair annihilation in QED – continued [13P]

We continue with the process of pair annihilation from the last exercise sheet. You showed that the matrix element can be written as

$$\mathcal{M} = -ie^2 \epsilon_\mu^{(\lambda_1)*}(k_1) \epsilon_\nu^{(\lambda_2)*}(k_2) \bar{v}(p_2, s_2) \left[\frac{\gamma^\nu \not{k}_1 \gamma^\mu - 2p_1^\mu \gamma^\nu}{2p_1 \cdot k_1} + \frac{\gamma^\mu \not{k}_2 \gamma^\nu - 2p_1^\nu \gamma^\mu}{2p_1 \cdot k_2} \right] u(p_1, s_1) \,. \tag{1.1}$$

Here, we denote by λ_1 and λ_2 the photon polarisations, and by s_1 and s_2 the electron and positron spins.

In order to compute a cross section, we now need the squared matrix element, averaged over all initial-state spins s_1, s_2 and summed over all final-state polarisations λ_1, λ_2. It is:

$$\begin{aligned} \frac{1}{4} \sum_{\lambda_1, \lambda_2; s_1, s_2} |\mathcal{M}|^2 = 2e^4 \Bigg[& \frac{p_1 \cdot k_2}{p_1 \cdot k_1} + \frac{p_1 \cdot k_1}{p_1 \cdot k_2} \\ & + 2m^2 \left(\frac{1}{p_1 \cdot k_1} + \frac{1}{p_1 \cdot k_2} \right) - m^4 \left(\frac{1}{p_1 \cdot k_1} + \frac{1}{p_1 \cdot k_2} \right)^2 \Bigg] \\ = 2e^4 \Bigg[& \frac{m^2 - u}{m^2 - t} + \frac{m^2 - t}{m^2 - u} \\ & + 4m^2 \left(\frac{1}{m^2 - t} + \frac{1}{m^2 - u} \right) - 4m^4 \left(\frac{1}{m^2 - t} + \frac{1}{m^2 - u} \right)^2 \Bigg] , \end{aligned} \tag{1.2}$$

where in the second line, we replaced the scalar products of the momenta in terms of the Mandelstam variables $s = (k_1 + k_2)^2 = (p_1 + p_2)^2$, $t = (k_1 - p_1)^2 = (k_2 - p_2)^2$, $u = (k_1 - p_2)^2 = (k_2 - p_1)^2$, as introduced on the previous sheet. m denotes the electron and positron mass. Note that the prefactor of $\frac{1}{4}$ originates from averaging over the initial-state spins. In the following, we want to calculate this squared matrix element step-by-step.

(a) **[2P]** First, compute $\mathcal{M}^\dagger$, which is the conjugate of Eq. (1.1). Then, write down $|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^\dagger$ and sum over all photon polarisations. Do not sum yet over the electron and positron spins.

Hint: You may use for the polarisation sum the following simplified expression:

$$\sum_\lambda \epsilon_\mu^{(\lambda)*}(k)\, \epsilon_\rho^{(\lambda)}(k) \to -g_{\mu\rho} \,.$$

We sum over the physical polarisations only. Do you have an idea why we are able to simplify the polarisation sum, i.e. why we can neglect the k-dependent terms on the right-hand side?

(b) **[6P]** Now, evaluate the sum over initial-state spins. You should obtain, with your result from part (a), the following expression:

$$\frac{1}{4}\sum_{\lambda_1,\lambda_2;s_1,s_2}|\mathcal{M}|^2 = \frac{e^4}{4}\left(\frac{A}{(2p_1\cdot k_1)^2} + \frac{B+C}{(2p_1\cdot k_1)(2p_1\cdot k_2)} + \frac{D}{(2p_1\cdot k_2)^2}\right),$$

where A, B, C, and D are expressions for traces over Dirac matrices:

$$\begin{aligned}
A &= \mathrm{Tr}\left[(\not{p}_2 - m)\left(\gamma^\nu \not{k}_1 \gamma^\mu - 2\gamma^\nu p_1^\mu\right)(\not{p}_1 + m)\left(\gamma_\mu \not{k}_1 \gamma_\nu - 2\gamma_\nu p_{1,\mu}\right)\right],\\
B &= \mathrm{Tr}\left[(\not{p}_2 - m)\left(\gamma^\nu \not{k}_1 \gamma^\mu - 2\gamma^\nu p_1^\mu\right)(\not{p}_1 + m)\left(\gamma_\nu \not{k}_2 \gamma_\mu - 2\gamma_\mu p_{1,\nu}\right)\right],\\
C &= \mathrm{Tr}\left[(\not{p}_2 - m)\left(\gamma^\mu \not{k}_2 \gamma^\nu - 2\gamma^\mu p_1^\nu\right)(\not{p}_1 + m)\left(\gamma_\mu \not{k}_1 \gamma_\nu - 2\gamma_\nu p_{1,\mu}\right)\right],\\
D &= \mathrm{Tr}\left[(\not{p}_2 - m)\left(\gamma^\mu \not{k}_2 \gamma^\nu - 2\gamma^\mu p_1^\nu\right)(\not{p}_1 + m)\left(\gamma_\nu \not{k}_2 \gamma_\mu - 2\gamma_\mu p_{1,\nu}\right)\right].
\end{aligned}$$

Evaluate the traces, using the known relations for the Dirac matrices as e.g. shown on exercise sheet 0.

Hint: You may use $\not{a}\not{b} = -\not{b}\not{a} + 2a\cdot b$, which implies $\not{a}\not{a} = a^2$, as well as

$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma_\mu = -2\gamma^\gamma\gamma^\beta\gamma^\alpha,$$

which was not shown on sheet 0. With these relations, you should be able to simplify the traces to only contain two Dirac matrices.

We now want to calculate the differential cross section,

$$\mathrm{d}\sigma = F\,\overline{\sum}|\mathcal{M}|^2\,\mathrm{dLIPS}_2$$

where

$$F = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2 m_2^2}}$$

is the flux factor with $m_1 = m_2 = m$ in our case, $\overline{\sum}|\mathcal{M}|^2$ is the averaged and summed squared matrix element of Eq. (1.2), and dLIPS_2 is the Lorentz-invariant 2-particle phase-space element:

$$\mathrm{dLIPS}_2 = \frac{\mathrm{d}^3k_1}{(2\pi)^3 2k_1^0}\frac{\mathrm{d}^3k_2}{(2\pi)^3 2k_2^0}(2\pi)^4\delta^{(4)}(p_1 + p_2 - k_1 - k_2).$$

(c) **[3P]** Evaluate the phase-space element in the centre-of-mass frame. Show that it can be simplified to:

$$\mathrm{dLIPS}_2 = \frac{1}{16\pi}\mathrm{d}\cos\theta,$$

where θ is defined as the angle between the momenta $\vec{p}_1$ and $\vec{k}_1$, i.e. $\vec{p}_1\cdot\vec{k}_1 = |\vec{p}_1||\vec{k}_1|\cos\theta$. After plugging in the flux factor, you should obtain for your differential cross section:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{1}{32\pi s\sqrt{1 - \frac{4m^2}{s}}}\overline{\sum}|\mathcal{M}|^2.$$

(d) **[2P]** We finally want to discuss the high-energy behaviour, $s \gg m^2$, of the obtained result. Show that in this limit, the cross section becomes:

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta}\right|_{s\gg m^2} = \frac{2\pi\alpha^2}{s}\frac{1+\cos\theta^2}{\sin\theta^2}\,,$$

where $\cos\theta$ is defined in the centre-of-mass frame as in part (c) and $\alpha = \frac{e^2}{4\pi}$ is the fine-structure constant.

Hint: Work out the behaviour of the Mandelstam variables t and u for $s \gg m^2$ (consider θ sufficiently far away from 0 and π).

Exercise 2: n-body final states **[7P]**

We want to discuss a general $2 \to n$ scattering process,

$$k_1 + k_2 \to p_1 + \ldots + p_n\,,$$

where k_1 and k_2 denote the initial-state momenta, and $p_1, \ldots, p_n$ denote the final-state momenta. The general expression for the differential cross section $\mathrm{d}\sigma$ was given in the previous exercise as:

$$\mathrm{d}\sigma = F\,\overline{\sum}|\mathcal{M}|^2\,\mathrm{dLIPS}_n,$$

In this exercise, we are interested in the phase-space element for n particles, which is defined as:

$$\mathrm{dLIPS}_n(P;p_1,\ldots,p_n) := (2\pi)^4\delta^{(4)}\left(\sum_{j=1}^{n} p_j - P\right)\prod_{j=1}^{n}\frac{\mathrm{d}^3 p_j}{(2\pi)^3 2p_j^0}$$

with the total four-momentum $P = k_1 + k_2$ given as the sum over the initial-state momenta.

(a) **[3P]** Show that dLIPS_3, the phase-space element for 3 final-state particles, can be written in terms of the 2-particle phase space:

$$\mathrm{dLIPS}_3(P;p_1,p_2,p_3) = \int\frac{\mathrm{d}M^2}{2\pi}\,\mathrm{dLIPS}_2(P;p_1,q)\,\mathrm{dLIPS}_2(q;p_2,p_3),$$

where q denotes a dummy four-momentum (i.e. one that does not correspond to any external particle) and $M^2 = q^2$ is its squared invariant mass.

Hint: Insert identities of the type $1 = \int \mathrm{d}^4 q\,\delta^{(4)}(q-q')$ and $1 = \int \mathrm{d}M^2\,\delta(M^2-q^2)$ in an intelligent way.

(b) **[2P]** Show that dLIPS_4 can be written as:

$$\mathrm{dLIPS}_4(P;p_1,p_2,p_3,p_4) = \\ \int\frac{\mathrm{d}M^2}{2\pi}\int\frac{\mathrm{d}M'^2}{2\pi}\,\mathrm{dLIPS}_2(P;q,q')\,\mathrm{dLIPS}_2(q;p_1,p_2)\,\mathrm{dLIPS}_2(q';p_3,p_4)$$

where q and M^2 are as defined in part (a) and q' and $M'^2 = q'^2$ are defined analogously.

(c) **[2P]** Show that dLIPS_n can in general be written as:

$$\mathrm{dLIPS}_n(P;p_1,\ldots,p_n) = \int\frac{\mathrm{d}M^2}{2\pi}\,\mathrm{dLIPS}_2(P;p_1,q)\,\mathrm{dLIPS}_{n-1}(q;p_2,\ldots,p_n)$$

where q and M^2 are as defined part (a).