Exercise Sheet 12

Exercises:
Dr. Christoph Borschensky (`christoph.borschensky@kit.edu`)
Dr. Francisco Arco (`francisco.arco@kit.edu`)
M.Sc. Felix Egle (`felix.egle@kit.edu`)

Issued: Mon, 29.01.24
Hand-in Deadline: Mon, 05.02.24
Discussion: Wed/Thu, 07/08.02.24

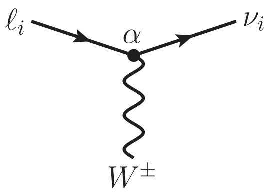
Exercise 1: Muon decay [20P]

We want to calculate the partial decay width and thereby lifetime of a muon μ^-. The muon is part of the second generation of leptons, and its interactions are thus very similar to the ones of the electron. The muon is just significantly heavier (approximately 200 times heavier than an electron), and it is unstable and decays into lighter particles. Its dominant decay mode is

$$\mu^-(p_1) \to \nu_\mu(p_2) + e^-(p_3) + \bar{\nu}_e(p_4)\,, \tag{1.1}$$

where ν_μ and $\bar{\nu}_e$ in the final state denote the muon neutrino and the electron anti-neutrino, respectively. As the name suggests, neutrinos are electrically neutral, but they carry a flavour quantum number, hence we distinguish between an electron neutrino and a muon neutrino.

We know that in QED, photons only couple to charged particles. Thus, this process is not mediated by QED, but by the *weak interaction*, and more precisely by the exchange of the electrically charged vector boson $W^\pm$ with mass M_W. The Feynman rule for the coupling of a lepton and its corresponding neutrino of the same flavour to the $W^\pm$ boson is given by:



$$= i\frac{g}{\sqrt{2}}\gamma^\alpha\omega_-\,,$$

where g denotes the coupling constant of the weak interaction, $i = e, \mu$ denotes the flavour index of the lepton ℓ and the neutrino ν, and ω_- is the left-chiral projector (which we introduced together with ω_+ on exercise sheet 2, $\omega_\mp = \frac{1}{2}(1 \mp \gamma^5)$).

In the following, we assume that the neutrinos are massless $m_{\nu_e} = m_{\nu_\mu} = 0$, while the electron and muon masses are given by m_e and m_μ, respectively.

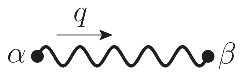
(a) **[1P]** Draw the Feynman diagram for the decay process of Eq. (1.1), and add all momentum labels for external and internal lines as well as the Lorentz indices for the vertices.

(b) **[2P]** Calculate the corresponding matrix element $\mathcal{M}$ and show that it simplifies to

$$\mathcal{M} = -i2\sqrt{2}G_F\,\bar{u}(p_2,s_2)\gamma^\alpha\omega_-u(p_1,s_1)\,\bar{u}(p_3,s_3)\gamma_\alpha\omega_-v(p_4,s_4)$$

after assuming typical momenta $q^2 \simeq m_\mu^2 \ll M_W^2$ where $q = p_1 - p_2$ is the momentum flowing through the $W^\pm$ boson propagator, and where G_F is the Fermi constant defined as $G_F = \frac{\sqrt{2}g^2}{8M_W^2}$. The s_i with $i = 1, \ldots, 4$ denote the spins of the fermions.

Hint: The Feynman rule for the $W^{\pm}$-boson propagator in the Feynman gauge is:



$$= \frac{-i g_{\alpha\beta}}{q^2 - M_W^2}\,.$$

(c) **[2P]** Determine the conjugated matrix element $\mathcal{M}^\dagger$ and simplify it as far as possible, using the same assumptions as in part (b).

Hint: Using the properties of the Dirac matrices, first show that

$$\begin{aligned}
[\bar{u}(p_2,s_2)\gamma^\alpha \omega_- u(p_1,s_1)]^\dagger &= \bar{u}(p_1,s_1)\gamma^\alpha \omega_- u(p_2,s_2)\,,\\
[\bar{u}(p_3,s_3)\gamma_\alpha \omega_- v(p_4,s_4)]^\dagger &= \bar{v}(p_4,s_4)\gamma_\alpha \omega_- u(p_3,s_3)\,.
\end{aligned}$$

(d) **[6P]** Let us now evaluate the squared matrix element after averaging over the initial-state spin and summing over all final-state spin configurations:

$$\overline{|\mathcal{M}|^2} = \frac{1}{2}\sum_{s_1,s_2,s_3,s_4} \mathcal{M}\mathcal{M}^\dagger\,,$$

where the overline denotes the averaging and summing. As an intermediate step, you should obtain:

$$\overline{|\mathcal{M}|^2} = 4G_F^2\,\mathrm{Tr}\left[\not{p}_2\gamma^\alpha\omega_-(\not{p}_1 + m_\mu)\gamma^\beta\omega_-\right]\mathrm{Tr}\left[\not{p}_4\gamma_\beta\omega_-(\not{p}_3 + m_e)\gamma_\alpha\omega_-\right],$$

where you have to evaluate the two traces over the Dirac matrices. Finally, after properly contracting all Lorentz indices, show that the expression simplifies to

$$\overline{|\mathcal{M}|^2} = 64G_F^2(p_1\cdot p_4)(p_2\cdot p_3)\,.$$

Hint: Remind that

$$\omega_\mp^2 = \omega_\mp\,,\qquad \omega_-\omega_+ = 0\,,\qquad \gamma^\mu\omega_- = \omega_+\gamma^\mu\,,$$

and use in addition to the trace relations of sheet 0 the following relations:

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma}\,,\qquad \epsilon^{\alpha\beta\nu\mu}\epsilon_{\alpha\beta}{}^{\rho\sigma} = -2\left(g^{\nu\rho}g^{\mu\sigma} - g^{\nu\sigma}g^{\mu\rho}\right),$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita symbol.

With the squared matrix element, we can now calculate the partial decay width of the muon:

$$\mathrm{d}\Gamma = \frac{1}{2E_1}\overline{|\mathcal{M}|^2}\mathrm{d}\Phi_3 \quad\text{with}\quad \mathrm{d}\Phi_3 = \prod_{i=2}^{4}\frac{\mathrm{d}^3 p_i}{(2\pi)^3 2E_i}(2\pi)^4\delta^{(4)}(p_1 - p_2 - p_3 - p_4)\,,$$

where the flux factor now only depends on the energy of the decaying particle, E_1. In the following, we work in the rest frame of the muon, i.e. $p_1 = (m_\mu, \vec{0})^T$, so that $E_1 = m_\mu$, and, as $m_e \ll m_\mu$, it is a good approximation to consider the electron as massless also, $m_e = 0$. As there is only one non-zero mass appearing from now on, let us write for simplicity $m = m_\mu$.

(e) **[2P]** First, show that $\overline{|\mathcal{M}|^2}$ from part (d) can be expressed only in terms of m and E_4, where E_4 is the energy of $\bar{\nu}_e$.

(f) **[2P]** With

$$\frac{\mathrm{d}^3 p_2}{2E_2} = \mathrm{d}^4 p_2\, \theta(E_2)\delta(p_2^2)\,, \qquad \text{where, as usual, } \theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases},$$

you can evaluate the $\mathrm{d}^4 p_2$ integration using the δ-distribution. Then, show that

$$\delta(p_2^2) = \frac{1}{2E_3 E_4}\delta\left(\cos\vartheta_4 - C\right) \quad \text{with} \quad C = 1 + \frac{m^2 - 2m(E_3 + E_4)}{2E_3 E_4}\,,$$

where ϑ_4 denotes the angle between $\vec{p}_3$ and $\vec{p}_4$, i.e. $\vec{p}_3 \cdot \vec{p}_4 = |\vec{p}_3||\vec{p}_4| \cos\vartheta_4$.

(g) **[2P]** Show that the three-particle phase space integrated over the angles associated with e^- and $\bar{\nu}_e$ is given by

$$\mathrm{d}\Phi_3 = \frac{1}{4(2\pi)^3}\mathrm{d}E_3\, \mathrm{d}E_4\, \theta(m - E_3 - E_4)\, \theta(1 - C)\, \theta(1 + C)\,.$$

Use the θ-functions to show that the integration boundaries of E_3 and E_4 can be chosen as

$$E_3 \in [0, \frac{m}{2}] \qquad \text{and} \qquad E_4 \in [\frac{m}{2} - E_3, \frac{m}{2}]\,.$$

Hint: Rewrite the remaining integration measures into an energy-dependent and an angular part,

$$\mathrm{d}^3 p_i = E_i^2\, \mathrm{d}E_i\, \mathrm{d}\Omega_i \qquad \text{with} \qquad \mathrm{d}\Omega_i = \mathrm{d}\phi_i\, \mathrm{d}\cos\vartheta_i\,,$$

where the $\mathrm{d}\phi_i$ integrations are over the interval $[0, 2\pi]$ and the $\mathrm{d}\cos\vartheta_i$ integrations are over the interval $[-1, 1]$. Note that you can choose the reference axis of the angles arbitrarily. Using your knowledge about $\overline{|\mathcal{M}|^2}$, which of the angular integrations can you then immediately perform?

(h) **[2P]** With the phase-space element of part (g) only depending on the two energies E_3 and E_4, evaluate first the integration over the energy of the electron anti-neutrino, E_4, to obtain the energy spectrum of the electron $\mathrm{d}\Gamma/\mathrm{d}E_3$. Then, integrate over E_3 to show that you obtain for the partial decay width the expression as shown in the lecture:

$$\Gamma = \frac{m^5 G_F^2}{192\pi^3}\,.$$

(i) **[1P]** Determine the life time $\tau_\mu = \hbar/\Gamma$ of the muon and compare with the experimental value. The numerical parameters are (see Review of Particle Physics)

$$\begin{aligned} m = m_\mu &= 0.1056583755\,\mathrm{GeV}\,, \\ G_F &= 1.1663788 \cdot 10^{-5}\,\mathrm{GeV}^{-2}\,, \\ \hbar &= 6.582119569 \cdot 10^{-25}\,\mathrm{GeV\,s}\,, \\ \tau_\mu^{\mathrm{exp.}} &= 2.1969811 \cdot 10^{-6}\,\mathrm{s}\,. \end{aligned}$$