Theoretical Particle Physics I

Prof. Dr. Milada Margarete Mühlleitner **WS 2023/24**

Exercise Sheet 13

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This last exercise sheet consists of bonus exercises which do not count towards the required amount of points. However, if you are still missing points for the required 50% for sheets 1–6 or sheets 7–12, you may obtain them here.

Exercise 1: Compton scattering [bonus 6P]

On exercise sheet 11 we calculated the differential cross section for the annihilation of an electron-postrion pair. For the squared and averaged matrix element we obtained

$$\begin{aligned}\overline{|\mathcal{M}_{ee\to\gamma\gamma}|^2} = 2e^4\Bigg[&\frac{m^2-u}{m^2-t}+\frac{m^2-t}{m^2-u}\\ &+4m^2\left(\frac{1}{m^2-t}+\frac{1}{m^2-u}\right)-4m^4\left(\frac{1}{m^2-t}+\frac{1}{m^2-u}\right)^2\Bigg],\end{aligned} \tag{1.1}$$

where m is the electron mass and the Mandelstam variables are defined as on sheet 11. We now want to deduce the squared matrix element for another process, namely Compton scattering:

$$e^-(p_1)+\gamma(k_1)\longrightarrow e^-(p_2)+\gamma(k_2)\,.$$

We label the momenta again as p_i for the electrons and k_j for the photons $(i,j=1,2)$, with momentum conservation telling us that $p_1+k_1=p_2+k_2$.

The Feynman diagrams for Compton scattering are very similar to the ones for pair annihilation, and can be obtained by a suitable rearrangement of the external legs. This method of connecting the two processes goes by the name of *crossing symmetry.*

(a) **[2P]** First, find the *crossing relations*, i.e. the replacement rules for the momenta to go from pair annihilation to Compton scattering. What are the replacement rules for the Mandelstam variables? Finally, use the crossing symmetry to write down the squared matrix element for Compton scattering immediately from the pair annihilation one of Eq. (1.1).

Hint: You will obtain an overall minus sign that you have to remove. Do you have an idea why it occurs?

Note: The crossing symmetry only works to obtain the squared matrix element. The phase-space factor has to be recalculated, and thus the cross section will be different.

As the result from part (a) is Lorentz-invariant, we can choose our reference frame as it fits us best. The next part of the calculation will thus be carried out in the so-called "lab(oratory) frame" where the electron in the initial state is at rest and the photon in the initial state moves along the positive z direction. We therefore have for their four-momenta

$$p_1 = (m, \vec{0})^T \,, \qquad k_1 = (\omega_1, 0, 0, \omega_1)^T \,,$$

with the ingoing photon energy ω_1. We furthermore choose our coordinate system such that the four-momentum of the outgoing photon is only in the y-z plane,

$$k_2 = (\omega_2, 0, \omega_2 \sin\theta, \omega_2 \cos\theta)^T \,,$$

where θ is the scattering angle with respect to the direction of the ingoing photon, and ω_2 is the energy of the outgoing photon.

(b) **[1P]** Show by using momentum conservation that ω_2 can be expressed in terms of ω_1 and $\cos\theta$ as

$$\omega_2 = \frac{\omega_1}{1 + \frac{\omega_1}{m}(1 - \cos\theta)} \,.$$

(c) **[2P]** We now want to compute the differential cross-section $\mathrm{d}\sigma$ in the lab frame. Show that you can write it as

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{1}{32\pi m^2} \frac{\omega_2^2}{\omega_1^2} \overline{|\mathcal{M}|^2} \,, \tag{1.2}$$

by calculating the phase space and the flux factor in the lab frame. Then, insert your result for the squared matrix element from part (a) to obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{m^2} \left(\frac{\omega_2}{\omega_1}\right)^2 \left(\frac{\omega_2}{\omega_1} + \frac{\omega_1}{\omega_2} - \sin^2\theta\right) \,,$$

where as before $e^2 = 4\pi\alpha$.

(d) **[1P]** Finally, we want to study the *Thomson limit*, i.e. the low-energy limit $\omega_1 \to 0$. Show that you find the classical result

$$\sigma_{\mathrm{tot}} = \int \mathrm{d}\cos\theta \, \frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{8\pi\alpha^2}{3m^2} \,.$$

Exercise 2: Scalar QED [bonus 14P]

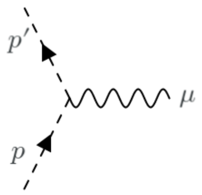
In the lecture and the previous exercises, we have only discussed QED involving fermions, i.e. the interactions between electrons and photons, as it is part of the Standard Model of particle physics. If we go *beyond the Standard Model*, we can come across new hypothetical particles that carry an electromagnetic charge. One such possible extension to the Standard Model is *Supersymmetry* (SUSY), which adds one or more partners to every Standard Model particle with similar properties, other than their spin quantum number. The partner of the electron would thus be the scalar electron, or *selectron* $\tilde{e}$. SUSY thus serves as a good motivation to analyse QED involving scalar particles.

As the selectron carries the same charge and has the same mass m as the electron, it is described by the following Lagrangian for a complex scalar field:

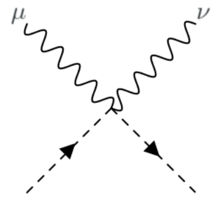
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D^\mu \tilde{e})^\dagger (D_\mu \tilde{e}) - m^2 \tilde{e}^\dagger \tilde{e}\,, \tag{2.1}$$

with the usual expression for the field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the covariant derivative $D_\mu = \partial_\mu + ieA_\mu$ leading to an interaction between the selectron $\tilde{e}$ and the photon A_μ.

(a) **[4P]** Using the Lagrangian of Eq. (2.1), derive the momentum-space Feynman rules for the interaction between photons and selectrons using Wick's theorem, and show that



$$= -ie(p+p')^\mu,$$



$$= 2ie^2 g^{\mu\nu},\,, \tag{2.2}$$

where p is defined as ingoing and p' as outgoing.

Hint: Remind what we did to derive Feynman rules on exercise sheet 10. Choose your initial and final states appropriately for each interaction vertex. As a start, first, plug the covariant derivative into Eq. (2.1) to obtain the two interactions. One of the terms comes with a derivative of the selectron field. Think about what the derivative will turn into in momentum space.

(b) **[2P]** We now want to look at Compton scattering of selectrons,

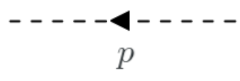
$$\tilde{e}(p_1) + \gamma(k_1) \longrightarrow \tilde{e}(p_2) + \gamma(k_2)$$

with p_1, k_1 being the ingoing and p_2, k_2 the outgoing momenta of the selectron and photon, respectively. Draw all Feynman diagrams (there are 3 in total) and write down the full matrix element, using the Feynman rules given above. You should obtain

$$\begin{aligned}\mathcal{M} = e^2 \Big[2g^{\mu\nu} &- \frac{1}{s-m^2}(2p_1+k_1)^\mu (2p_2+k_2)^\nu \\ &- \frac{1}{u-m^2}(2p_1-k_2)^\nu (2p_2-k_1)^\mu \Big] \epsilon_\mu^{(\lambda_1)}(k_1) \epsilon_\nu^{(\lambda_2)*}(k_2)\,,\end{aligned}$$

with the Mandelstam variables $s = (p_1+k_1)^2$ and $u = (p_1-k_2)^2$.

Hint: Use the following expressions for the selectron propagator:



$$= \frac{i}{p^2 - m^2}\,.$$

Remind that scalar fields, unlike fermionic or vector ones, do not have any external wave-function factors.

(c) **[2P]** Show that the matrix element obtained in part (b) fulfills the Ward identity, i.e. replace one of the polarisation vectors with its corresponding momentum and show that the resulting expression vanishes.

(d) **[3P]** We now want to calculate the differential cross section. We can reuse the expression of Eq. (1.2) from the Compton scattering exercise above, as the phase space and the flux factor are the same as in the fermonic case, and thus only need to calculate the squared matrix element. As selectrons do not have spin, we only need to sum/average over the polarisations of the photons. Show that the squared and averaged matrix element can be written as

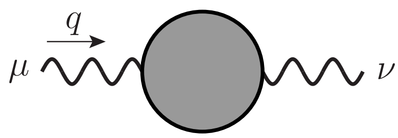
$$\overline{|\mathcal{M}|^2} = 4e^4 \left[1 + 2m^2 \left(\frac{1}{s-m^2} - \frac{1}{m^2-u} \right) + 2m^4 \left(\frac{1}{s-m^2} - \frac{1}{m^2-u} \right)^2 \right] .$$

Then, using the same parametrisation for the momenta as in exercise 1, show that the differential cross section becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\cos\theta} = \frac{\pi\alpha^2}{m^2} \frac{\omega_2^2}{\omega_1^2} (1 + \cos\theta^2) .$$

(e) **[1P]** As in the case for fermions, go into the limit of vanishing photon energy $\omega_1 \to 0$, and compare your result with the one for an electron from the previous exercise.

(f) **[2P]** The above calculation took into account only tree-level diagrams at $\mathcal{O}(e^2)$. At higher orders in perturbation theory, one-loop diagrams have to be included. Some of these one-loop diagrams will involve as a subdiagram the vacuum polarisation,



as discussed also in the lecture. The grey circle here stands for all possible one-loop insertions. Draw all diagrams for the vacuum polarisation in scalar QED. Then, using the Feynman rules from above, write down the corresponding matrix elements (you do not have to evaluate the integrals). Consider the external photon propagators as *amputated*, i.e. you do not have to write them down.